MATHEMATICS

THE ROUTH-HURWITZ CONDITION FOR THE BIQUADRATIC EQUATION

BY

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1. In order to answer the question whether the equilibrium of a mechanical system of n degrees of freedom is stable we have to write down the equation (of degree $2n$) for the frequencies of its small oscillations. Leaving out of account the cases of equal roots it is necessary and sufficient for stability that all roots have a non-positive real part.

The conditions for an algebraic equation with real coefficients having its roots on the lefthand-side L of the complex plane have long ago been given by ROUTH and HURWITZ and consist of a system of inequalities for certain determinants. The proof of the general theorem being rather elaborate, we here give an elementary proof for the case $n=2$ and we add some remarks on a somewhat surprising discontinuity which arises in the matter.

2. It is easily seen that the quadratic equation

$$x^2+px+q=0$$

has two roots with negative real parts if and only if $p>0$, $q>0$. If one root is on the i-axis and the other one left of it we have $p>0$, $q=0$; for two roots on this axis $p=0$, $q>0$; if both roots are on the righthand-side R we have $p<0$, $q>0$; all these conditions are necessary and sufficient. We now consider the equation of the fourth degree

$$Q \equiv x^4+a_1x^3+a_2x^2+a_3x+a_4=0. \tag{1}$$

It is always possible to write, in at least one way, the lefthand-side as the product of two quadratic forms with real coefficients,

$$Q \equiv (x^2+p_1x+q_1)(x^2+p_2x+q_2) \tag{2}$$

hence

$$a_1=p_1+p_2,\ a_2=p_1p_2+q_1+q_2,\ a_3=p_1q_2+p_2q_1,\ a_4=q_1q_2. \tag{3}$$

For all the roots of (1) to be in L it is obviously necessary and sufficient that p_i and q_i are positive. Therefore in view of (3) we have: *a necessary condition for the roots of* (1) *having negative real parts is* $a_i>0$ ($i=1, 2, 3, 4$). This system of conditions however is not sufficient, as the example

$(x^2-x+2)(x^2+2x+3)=x^4+x^3+3x^2+x+6$ shows. But if $a_i>0$ it is not possible that either one root of three roots lies in L (for then $a_4 \leqslant 0$); it is also impossible that no root is in it (for then $a_1 \leqslant 0$). Hence if $a_i>0$ at least two roots are in L; the other ones are either both in L, or both on the i-axis, or both in R. In order to distinguish between these cases we deduce the condition for two roots being on the i-axis. If yi (y real, $\neq 0$) is a root, then

$$y^4-a_2y^2+a_4=0, \quad -a_1y^2+a_3=0.$$

Hence

$$C \equiv a_1^2a_4+a_3^2-a_1a_2a_3=0. \tag{4}$$

Now by means of (3) we have

$$\left\{\begin{aligned} C &\equiv a_1^2q_1q_2+a_3^2-a_1a_3(p_1p_2+q_1+q_2)=a_1^2q_1q_2-a_1a_3(q_1+q_2)+a_3^2-p_1p_2a_1a_3 \\ &=(a_1q_1-a_3)(a_1q_2-a_3)-p_1p_2a_1a_3=-p_1p_2\,\{a_1a_3+(q_1-q_2)^2\}. \end{aligned}\right. \tag{5}$$

In view of $a_1>0$, $a_3>0$ the second factor is positive; furthermore $a_1=p_1+p_2>0$ hence p_1 and p_2 cannot both be negative. Therefore $C<0$ implies $p_1>0$, $p_2>0$, for $C=0$ we have either $p_1=0$ or $p_2=0$ (and not both, because $a_3>0$), for $C>0$ p_1 and p_2 have different signs. We see from (2) that all roots of (1) are in L if p_1 and p_2 are positive.

Hence: *a set of necessary and sufficient conditions for all roots of* (1) *to be on the lefthand-side of the complex plane is*

$$a_i>0 \ (i=1, 2, 3, 4), \ C<0. \tag{6}$$

The condition $C<0$ is a Routh–Hurwitz inequality.

3. We now proceed to the cases where all roots have non-positive real parts, so that they lie either in L or on the i-axis.

If three roots are in L and one on the imaginary axis the root must be $x=0$. Reasoning along the same lines as before we find that necessary and sufficient conditions for this are

$$a_i>0 \ (i=1, 2, 3), \ a_4=0, \ C<0. \tag{7}$$

If two roots are in L and two (different) roots on the i-axis we have $p_1>0$, $q_1>0$, $p_2=0$, $q_2>0$ and the conditions are

$$a_i>0 \ (i=1, 2, 3, 4), \ C=0. \tag{8}$$

If one root is in L and three are on the y-axis, then $p_1>0$, $q_1=0$, $p_2=0$, $q_2>0$ and the conditions are

$$a_i>0 \ (i=1, 2, 3), \ a_4=0, \ C=0. \tag{9}$$

The conditions (7), (8) and (9) are bordercases of (6). This does not occur with the last type we have to consider: all roots are on the i-axis. We now have $p_1=0$, $p_2=0$, $q_1>0$, $q_2>0$. Hence $a_2>0$, $a_4>0$, $a_1=a_3=0$ and therefore $C=0$. This set of relations is necessary, but not sufficient, as the example $Q \equiv x^4+6x^2+25=0$ (which has two roots in L and two in R) shows. The proof given above is not valid because as seen from (5) $C=0$ does not imply now $p_1p_2=0$, the second factor being zero for $a_1a_3=0$ and $q_1=q_2$. The condition can of course easily be given; the equation (1) is

$$x^4+a_2x^2+a_4=0$$

and therefore it reads

$$a_2>0,\ a_4>0,\ a_2^2>4a_4. \tag{10}$$

Summing up we have: all roots of (1) (assumed to be different) have non-positive real parts if and only if one of the two following sets of conditions is satisfied:

$$\begin{cases} \text{A. } a_1>0,\ a_2>0,\ a_3>0,\ a_4\geqslant 0,\ a_2\geqslant \dfrac{a_1^2a_4+a_3^2}{a_1a_3} \\ \text{B. } a_1=0,\ a_2>0,\ a_3=0,\ a_4>0,\ a_2>2\sqrt{a_4}. \end{cases} \tag{11}$$

4. One could expect B to be a limit of A, so that for $a_1\to 0$, $a_3\to 0$ the set A would continuously tend to B. That is not the case. Remark first of all that the roots of (1) never lie outside R if $a_1=0$, $a_3\neq 0$ (or $a_1\neq 0$, $a_3=0$). Furthermore, if A is satisfied and we take $a_1=b_1\varepsilon$, $a_3=b_3\varepsilon$, where b_1 and b_3 are fixed and $\varepsilon\to 0$, the last condition of A reads ($\varepsilon\neq 0$) $a_2\geqslant \dfrac{b_1^2a_4+b_3^2}{b_1b_3}=g_1$, while for $\varepsilon=0$ we have $a_2>2\sqrt{a_4}=g_2$. Obviously we have $g_1-g_2=\dfrac{(b_1\sqrt{a_4}-b_3)^2}{b_1b_3}$, so that $g_1>g_2$ but for $\dfrac{b_3}{b_1}=\sqrt{a_4}$. Here is the discontinuity we mentioned above. It plays a part in questions regarding the stability of equilibrium. The coefficients a_1 and a_3 depend on the linear damping forces and it is well known that the stability condition may change in a discontinuous way if a very small damping vanishes at all [1]). The phenomenon may be illustrated by a geometrical diagram. As $a_4>0$ we substitute in (1) $x=cy$, where c is the positive fourth root of a_4. The new equation reads

$$y^4+b_1y^3+b_2y^2+b_3y+1=0 \tag{12}$$

where $b_i=a_i/c^i$ $(i=1,2,3,4)$. If we substitute $a_i=c^ib_i$ in A en B we get the same condition as when we write b_i for a_i, which was to be expexted, because if the roots of (1) are outside R, those of (12) are also outside R and inversely. We can therefore restrict ourselves to the case $a_4=1$, so that we have only three parameters, a_1, a_2, a_3. We take them as coordinates in a orthogonal coordinate system (fig. 1). The condition $C=0$ or

$$a_1a_2a_3=a_1^2+a_3^2$$

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[1]) Ziegler, Die Stabilitätskriterien der Elastomechanik, Ing. Arch. **20**, 49–56 (1952); Bottema, On the stability of the equilibrium of a linear mechanical system, Journal of Appl. Math. Phys. (ZAMP), **6**, 97–104 (1955).

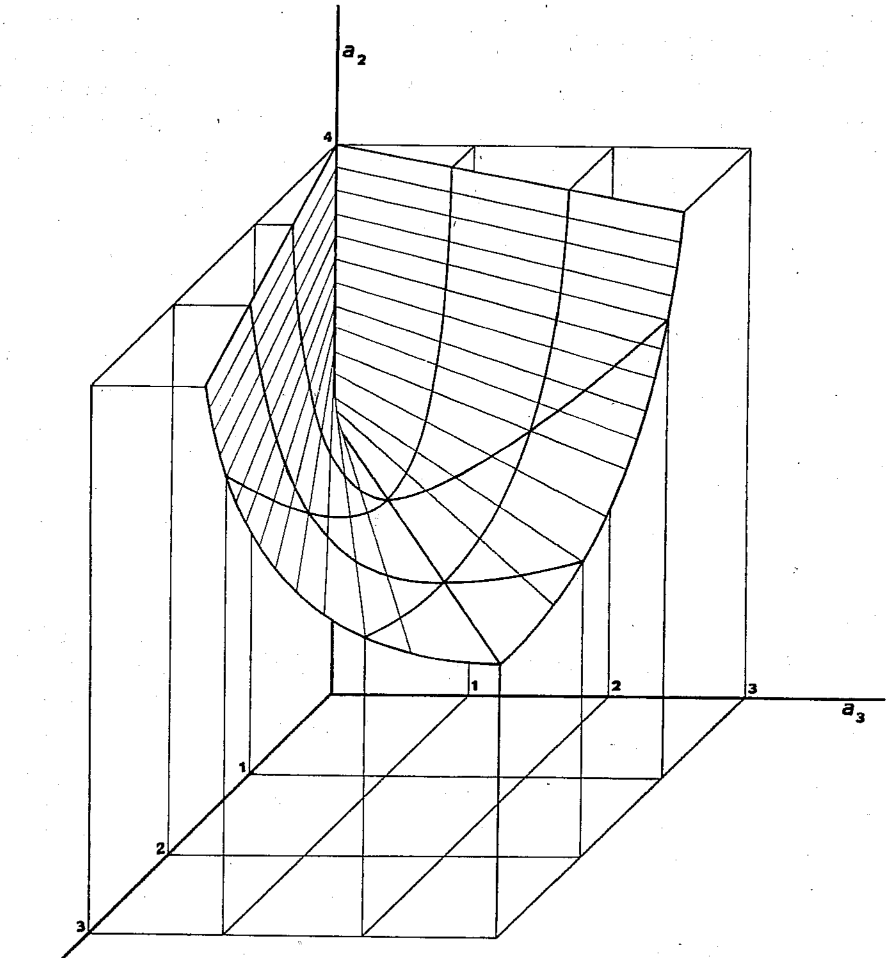

Fig. 1

is the equation of a surface V of the third degree, which we have to consider for $a_1 \geqslant 0$, $a_3 \geqslant 0$. Obviously V is a ruled surface, the line $a_3 = ma_1$, $a_2 = m + 1/m$ $(0 < m < \infty)$ being on V. The line is parallel to the $0a_1a_3$-plane and intersects the a_2-axis in $a_1 = a_3 = 0$, $a_2 = m + 1/m \geqslant 2$. The a_2-axis is the double line of V, $a_2 \geqslant 2$ being its active part. Two generators pass through each point of it; they coincide for $a_2 = 2$ $(m = 1)$, and for $a_2 \to \infty$ their directions tend to those of the a_1- and a_3-axis $(m = 0,\ m = \infty)$.

The conditions A and B express that the imagepoint (a_1, a_2, a_3) lies on V or above V. The point $(0, 2, 0)$ is on V, but if we go to the a_2-axis along the line $a_3 = ma_1$ the coordinate a_2 has the limit $m + 1/m$, which is > 2 but for $m = 1$.

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