THE ROUGH-HURWITZ CONDITION FOR THE BICOQUADRATIC EQUATION

BY

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1. In order to answer the question whether the equilibrium of a mechanical system of \( n \) degrees of freedom is stable we have to write down the equation (of degree \( 2n \)) for the frequencies of its small oscillations. Leaving out of account the cases of equal roots it is necessary and sufficient for stability that all roots have a non-positive real part.

The conditions for an algebraic equation with real coefficients having its roots on the lefthand-side \( L \) of the complex plane have long ago been given by Routh and Hurwitz and consist of a system of inequalities for certain determinants. The proof of the general theorem being rather elaborate, we here give an elementary proof for the case \( n=2 \) and we add some remarks on a somewhat surprising discontinuity which arises in the matter.

2. It is easily seen that the quadratic equation

\[
x^2 + px + q = 0
\]

has two roots with negative real parts if and only if \( p > 0, q > 0 \). If one root is on the \( i \)-axis and the other one left of it we have \( p > 0, q = 0 \); for two roots on this axis \( p = 0, q > 0 \); if both roots are on the righthand-side \( R \) we have \( p < 0, q > 0 \); all these conditions are necessary and sufficient.

We now consider the equation of the fourth degree

\[
Q = a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0.
\]

It is always possible to write, in at least one way, the lefthand-side as the product of two quadratic forms with real coefficients,

\[
Q = (x^2 + p_1x + q_1)(x^2 + p_2x + q_2)
\]

hence

\[
a_1 = p_1 + p_2, \quad a_2 = p_1q_2 + q_1 + q_2, \quad a_3 = p_1p_2 + p_2q_1 + q_1 + q_2, \quad a_4 = q_1q_2.
\]

For all the roots of (1) to be in \( L \) it is obviously necessary and sufficient that \( p_i \) and \( q_i \) are positive. Therefore in view of (3) we have: a necessary condition for the roots of (1) having negative real parts is \( a_i > 0 \) (\( i = 1, 2, 3, 4 \)). This system of conditions however is not sufficient, as the example
\[ (x^2 - x + 2)(x^2 + 2x + 3) = x^4 + x^3 + 3x^2 + x + 6 \] shows. But if \( a_i > 0 \) it is not possible that either one root of three roots lies in \( L \) (for then \( a_4 < 0 \)); it is also impossible that no root is in it (for then \( a_4 < 0 \)). Hence if \( a_i > 0 \) at least two roots are in \( L \); the other ones are either both in \( L \), or both on the \( i \)-axis, or both in \( R \). In order to distinguish between these cases we deduce the condition for two roots being on the \( i \)-axis. If \( y \) (\( y \) real, \( \neq 0 \)) is a root, then

\[ y^4 - a_2 y^2 + a_4 = 0, \quad -a_2 y^2 + a_4 = 0. \]

Hence

\[ C = a_2^2 a_4 + a_2^2 - a_2 a_4 a_3 = 0. \]

Now by means of (3) we have

\[ C = a_2^2 q_2 + a_2^2 - a_2 a_3 (p_1 p_2 + q_1 + q_2) - a_4 q_2 - a_2 a_3 (q_1 + q_2) + a_2^2 - p_1 p_2 a_3 a_5 \]

\[ = (a_2 q_1 - a_2) (a_2 q_2 - a_3) - p_1 p_2 a_3 a_5 = -p_1 p_2 (a_2 q_3 + (q_1 - q_2)^3). \]

In view of \( a_1 > 0 \), \( a_2 > 0 \) the second factor is positive; furthermore \( a_1 = p_1 = p_2 > 0 \) hence \( p_1 \) and \( p_2 \) cannot both be negative. Therefore \( C < 0 \) implies \( p_1 > 0, p_2 > 0 \), for \( C = 0 \) we have either \( p_1 = 0 \) or \( p_2 = 0 \) (and not both, because \( a_2 > 0 \)), for \( C > 0 \) \( p_1 \) and \( p_2 \) have different signs. We see from (2) that all roots of (1) are in \( L \) if \( p_1 \) and \( p_2 \) are positive.

Hence: a set of necessary and sufficient conditions for all roots of (1) to be on the left-hand-side of the complex plane is

\[ a_i > 0 \quad (i = 1, 2, 3, 4), \quad C < 0. \]

The condition \( C < 0 \) is a Routh–Hurwitz inequality.

3. We now proceed to the cases where all roots have non-positive real parts, so that they lie either in \( L \) or on the \( i \)-axis.

If three roots are in \( L \) and one on the imaginary axis the root must be \( x = 0 \). Reasoning along the same lines as before we find that necessary and sufficient conditions for this are

\[ a_i > 0 \quad (i = 1, 2, 3), \quad a_4 = 0, \quad C < 0. \]

If two roots are in \( L \) and two (different) roots on the \( i \)-axis we have \( p_1 > 0, q_1 > 0, \quad p_2 = 0, \quad q_2 > 0 \) and the conditions are

\[ a_i > 0 \quad (i = 1, 2, 3, 4), \quad C = 0. \]

If one root is in \( L \) and three are on the \( y \)-axis, then \( p_1 > 0, q_1 = 0, \quad p_2 = 0, \quad q_2 > 0 \) and the conditions are

\[ a_i > 0 \quad (i = 1, 2, 3), \quad a_4 = 0, \quad C = 0. \]

The conditions (7), (8) and (9) are bordercases of (6). This does not occur with the last type we have to consider: all roots are on the \( i \)-axis. We now have \( p_1 = 0, \quad p_2 = 0, \quad q_1 > 0, \quad q_2 > 0 \). Hence \( a_2 > 0, \quad a_4 > 0, \quad a_1 = a_3 = 0 \) and

\[ (x^2 - x + 2)(x^2 + 2x + 3) = x^4 + x^3 + 3x^2 + x + 6 \] shows: But if \( a_i > 0 \) it is not possible that either one root of three roots lies in \( L \) (for then \( a_4 < 0 \)); it is also impossible that no root is in it (for then \( a_4 < 0 \)). Hence if \( a_i > 0 \) at least two roots are in \( L \); the other ones are either both in \( L \), or both on the \( i \)-axis, or both in \( R \). In order to distinguish between these cases we deduce the condition for two roots being on the \( i \)-axis. If \( y \) (\( y \) real, \( \neq 0 \)) is a root, then

\[ y^4 - a_2 y^2 + a_4 = 0, \quad -a_2 y^2 + a_4 = 0. \]

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\[ = (a_2 q_1 - a_2) (a_2 q_2 - a_3) - p_1 p_2 a_3 a_5 = -p_1 p_2 (a_2 q_3 + (q_1 - q_2)^3). \]

In view of \( a_1 > 0 \), \( a_2 > 0 \) the second factor is positive; furthermore \( a_1 = p_1 = p_2 > 0 \) hence \( p_1 \) and \( p_2 \) cannot both be negative. Therefore \( C < 0 \) implies \( p_1 > 0, p_2 > 0, \) for \( C = 0 \) we have either \( p_1 = 0 \) or \( p_2 = 0 \) (and not both, because \( a_2 > 0 \)), for \( C > 0 \) \( p_1 \) and \( p_2 \) have different signs. We see from (2) that all roots of (1) are in \( L \) if \( p_1 \) and \( p_2 \) are positive.

Hence: a set of necessary and sufficient conditions for all roots of (1) to be on the left-hand-side of the complex plane is

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If one root is in \( L \) and three are on the \( y \)-axis, then \( p_1 > 0, q_1 = 0, \quad p_2 = 0, \quad q_2 > 0 \) and the conditions are

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The conditions (7), (8) and (9) are bordercases of (6). This does not occur with the last type we have to consider: all roots are on the \( i \)-axis. We now have \( p_1 = 0, \quad p_2 = 0, \quad q_1 > 0, \quad q_2 > 0 \). Hence \( a_2 > 0, \quad a_4 > 0, \quad a_1 = a_3 = 0 \) and
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therefore \( C = 0 \). This set of relations is necessary, but not sufficient, as the example \( Q \equiv x^4 + 6x^2 + 25 = 0 \) (which has two roots in \( L \) and two in \( R \)) shows. The proof given above is not valid because as seen from (5) \( C = 0 \) does not imply now \( p_1p_2 = 0 \), the second factor being zero for \( a_4a_3 = 0 \) and \( g_1 = g_2 \). The condition can of course easily be given; the equation (1) is

\[
x^4 + a_2x^2 + a_4 = 0
\]

and therefore it reads

(10) \[ a_2 > 0, \quad a_4 \geq 0, \quad a_3^2 \geq 4a_4. \]

Summing up we have: all roots of (1) (assumed to be different) have non-positive real parts if and only if one of the two following sets of conditions is satisfied:

\[
\begin{align*}
A: & \quad a_1 > 0, \quad a_2 > 0, \quad a_3 > 0, \quad a_4 > 0, \quad a_2a_3 = a_2 + a_3^2 \\
B: & \quad a_1 = 0, \quad a_2 > 0, \quad a_3 = 0, \quad a_4 > 0, \quad a_2 > 2\sqrt{a_4}.
\end{align*}
\]

4. One could expect \( B \) to be a limit of \( A \), so that for \( a_1 \to 0, a_2 \to 0 \) the set \( A \) would continuously tend to \( B \). That is not the case. Remark first of all that the roots of (1) never lie outside \( R \) if \( a_1 = 0, a_2 \neq 0 \) (or \( a_3 \neq 0, a_3 = 0 \)). Furthermore, if \( A \) is satisfied and we take \( a_1 = b_1\varepsilon, a_2 = b_2\varepsilon, \) where \( b_1 \) and \( b_2 \) are fixed and \( \varepsilon \to 0 \), the last condition of \( A \) reads \( (\varepsilon \neq 0) \)

\[
a_2 > \frac{2b_2^2 + b_1^2}{b_1b_2} = g_1, \quad \text{while for} \quad \varepsilon = 0 \quad \text{we have} \quad a_2 > 2\sqrt{a_4} = g_2.
\]

Obviously we have \( g_1 > g_2 \) but for \( \frac{b_2}{b_1} = \frac{\varepsilon}{\sqrt{a_4}} \). Here is the discontinuity we mentioned above. It plays a part in questions regarding the stability of equilibrium. The coefficients \( a_1 \) and \( a_3 \) depend on the linear damping forces and it is well known that the stability condition may change in a discontinuous way if a very small damping vanishes at all \(^1\). The phenomenon may be illustrated by a geometrical diagram. As \( a_4 > 0 \) we substitute in (1) \( x = cy \), where \( c \) is the positive fourth root of \( a_4 \). The new equation reads

(12) \[ g^4 + b_1g^3 + b_2g^2 + b_3g + 1 = 0 \]

where \( b_i = a_i/c \) (\( i = 1, 2, 3, 4 \)). If we substitute \( a_i = cb_i \) in \( A \) and \( B \) we get the same condition as when we write \( b_i \) for \( a_i \), which was to be expected, because if the roots of (1) are outside \( R \), those of (12) are also outside \( R \) and inversely. We can therefore restrict ourselves to the case \( a_4 = 1 \), so that we have only three parameters, \( a_1, a_2, a_3 \). We take them as coordinates in a orthogonal coordinate system (fig. 1). The condition \( C = 0 \) or

\[
a_1a_2a_3 = a_1^2 + a_3^2.
\]

is the equation of a surface $V$ of the third degree, which we have to consider for $a_1>0$, $a_3>0$. Obviously $V$ is a ruled surface, the line $a_3=ma_1$, $a_2=m+1/m$ ($0<m<\infty$) being on $V$. The line is parallel to the $\partial a_1a_3$-plane and intersects the $a_2$-axis in $a_1=a_2=0$, $a_2=m+1/m>2$. The $a_2$-axis is the double line of $V$, $a_2>2$ being its active part. Two generators pass through each point of it; they coincide for $a_2=2$ ($m=1$), and for $a_2 \to \infty$ their directions tend to those of the $\partial a_1$- and $\partial a_2$-axis ($m=0$, $m=\infty$).

The conditions $A$ and $B$ express that the imagepoint $(a_1, a_2, a_3)$ lies on $V$ or above $V$. The point $(0, 2, 0)$ is on $V$, but if we go to the $a_2$-axis along the line $a_3=ma_1$ the coordinate $a_3$ has the limit $m+1/m$, which is $>2$ but for $m=1$. 

1. of which

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If the

I: $t$

II: $t$

III: $t$

The relation may be written by $K$. (1) is represented by $\omega$. $V$ is the superficial section just mentioned.