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Collapse of Keldysh Chains and the Stability of Nonconservative Systems

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Non-self-adjoint operators arise in nonconservative problems of mechanics and physics. The theory of non-self-adjoint operators goes back to G. Birkhoff's works and was later developed by many scientists [1–7]. Keldysh [1] was the first to extend the concept of a Jordan chain of vectors to a wide class of non-self-adjoint operators. In the presence of parameters in the spectra of non-self-adjoint operators in the general position, there are multiple eigenvalues with Keldysh chains. It turns out that these eigenvalues determine the geometry of the boundary of the stability region for the corresponding nonconservative system. An effective technique for analyzing this boundary is the study of bifurcations of eigenvalues with varying parameters based on perturbation theory [2]. Such an analysis has been conducted only in the finite-dimensional case [8].

In this work, we consider eigenvalue problems for non-self-adjoint linear differential operators smoothly depending on a vector of real parameters. Explicit expressions are found that describe bifurcations of multiple eigenvalues with arbitrarily long Keldysh chains along smooth curves in the parameter space. The formulas derived use the eigen- and associated functions of the adjoint eigenvalue problems and the derivatives of the differential operator with respect to the parameters. As applications, we examine the boundaries of the stability regions of circulatory systems.

1. COLLAPSE OF A KELDYSH CHAIN

Consider the eigenvalue problem for a linear differential operator L [4]

$$l(u) = \lambda u, \quad U^s(u) = 0, \quad s = 1, 2, \ldots, m, \tag{1}$$

where

$$l(u) \equiv \sum_{i=0}^{m} a_i \frac{d^{m-i}u}{dx^{m-i}},$$

$$U^s(u) \equiv \sum_{i=0}^{m-1} \left(\alpha_i^s \frac{d^i u}{dx^i}\bigg|_{x=0} + \beta_i^s \frac{d^i u}{dx^i}\bigg|_{x=1} \right).$$

Assume that problem (1) is such that the coefficients of the differential expression $l(u)$ and the coefficients of the forms $U^s(u)$ are real functions smoothly depending on a parameter vector $\mathbf{p} \in \mathbb{R}^n$, i.e., C^∞ functions on an open set $\Omega \subset \mathbb{R}^n$. Suppose the spectrum of L at a point $\mathbf{p}_0$ contains a multiple eigenvalue λ_0 with a Keldysh chain of length k. This means that, at $\mathbf{p} = \mathbf{p}_0$, there exist an eigenfunction $u_0(x)$ and associated functions $u_1(x), \ldots, u_{k-1}(x)$ corresponding to λ_0 and satisfying the equations and boundary conditions

$$\begin{gathered} l_0(u_0) = \lambda_0 u_0, \quad U_0^s(u_0) = 0; \\ l_0(u_i) = \lambda_0 u_i + u_{i-1}, \quad U_0^s(u_i) = 0; \\ i = 1, 2, \ldots, k-1; \quad s = 1, 2, \ldots, m. \end{gathered} \tag{2}$$

For the adjoint eigenvalue problem [4], we have

$$\begin{gathered} l_0^*(v_0) = \bar{\lambda}_0 v_0, \quad V_0^s(v_0) = 0; \\ l_0^*(v_i) = \bar{\lambda}_0 v_i + v_{i-1}, \quad V_0^s(v_i) = 0; \\ i = 1, 2, \ldots, k-1; \quad s = 1, 2, \ldots, m. \end{gathered} \tag{3}$$

It follows from Eqs. (2) and (3) that the eigen- and associated functions of the adjoint problems are related by the equations

$$\begin{gathered} (u_j, v_0) = 0, \quad j = 0, 1, \ldots, k-2; \\ (u_{k-1}, v_0) \equiv (u_0, v_{k-1}) \neq 0; \end{gathered} \tag{4}$$

$$(u_{j-1}, v_i) \equiv (u_j, v_{i-1}), \quad i, j = 1, 2, \ldots, k-1, \tag{5}$$

where $(\varphi, \psi) = \int_0^1 \varphi(x)\overline{\psi(x)}dx$ is the scalar product of φ, ψ. The Keldysh chain is an analogue of the Jordan chain of vectors in eigenvalue problems for non-self-adjoint operators [1, 3–5, 7].

Suppose that a smooth curve issues from the point $\mathbf{p}_0$ in the parameter space in a direction $\mathbf{e} \in \mathbb{R}^n$ such that

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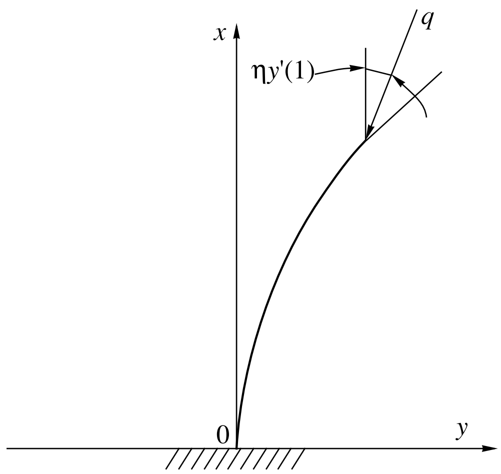

Fig. 1. Column loaded by a nonconservative force.

$\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon\mathbf{e}$ in the first approximation. Here, $\varepsilon \geq 0$ is a small parameter. Variations in the parameter vector lead to perturbations of the eigenvalues and eigenfunctions. In the case of a multiple eigenvalue with a Keldysh chain of length k, they are represented by series in fractional powers of the small parameter $\varepsilon^{j/k}$, $j = 0, 1, 2, \ldots$ [2]:

$$\lambda = \lambda_0 + \varepsilon^{\frac{1}{k}}\lambda_1 + \varepsilon^{\frac{2}{k}}\lambda_2 + \ldots,$$
$$u = u_0 + \varepsilon^{\frac{1}{k}}w_1 + \varepsilon^{\frac{2}{k}}w_2 + \ldots. \tag{6}$$

In this case, $l(u)$ and $U^s(u)$ take increments:

$$l(u) = l_0(u) + \varepsilon l_1(u) + \ldots,$$
$$U^s(u) = U_0^s(u) + \varepsilon U_1^s(u) + \ldots, \tag{7}$$

where $l_0 = l(u)|_{\mathbf{p} = \mathbf{p}_0}$, $U_0^s = U^s(u)|_{\mathbf{p} = \mathbf{p}_0}$, and $l_1(u)$ and $U_1^s(u)$ are given by

$$l_1(u) = \sum_{i=1}^{n} e_i \frac{\partial l}{\partial p_i}(u), \quad U_1^s(u) = \sum_{i=1}^{n} e_i \frac{\partial U^s}{\partial p_i}(u). \tag{8}$$

All derivatives in (8) are taken at the point $\mathbf{p}_0$. Thus, we consider regular perturbations whose order does not exceed the order of the unperturbed operator $L_0 = L(\mathbf{p}_0)$ [2].

Substituting (6) and (7) into Eqs. (1) and collecting the coefficients of equal powers of ε gives k boundary value problems for $w_1, w_2, \ldots, w_k$, which determine the first-order correction λ_1 to the eigenvalue λ_0:

$$\lambda_1 = \sqrt[k]{\langle \mathbf{f}_k, \mathbf{e}\rangle + i\langle \mathbf{g}_k, \mathbf{e}\rangle}. \tag{9}$$

Here, $\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{i=1}^{n} a_i b_i$ is the scalar product of vectors $\mathbf{a}$, $\mathbf{b} \in \mathbb{R}^n$ in the parameter space, and $\mathbf{f}_k$ and $\mathbf{g}_k$ are real vectors with components

$$f_k^j + i g_k^j = \frac{\left(\frac{\partial l}{\partial p_j}(u_0), v_0\right) - \sum_{s=1}^{m} \frac{\partial U^s}{\partial p_j}(u_0) V_0^{2m-s+1}(v_0)}{(u_{k-1}, v_0)}, \tag{10}$$

corresponding to the k-fold eigenvalue λ_0 at $\mathbf{p}_0$, and the forms $V_0^{m+1}, \ldots, V_0^{2m}$ are the coefficients of $U_0^m, \ldots, U_0^1$ in the Lagrange formula [4]

$$(l_0(u_0), v_0) - (u_0, l_0^*(v_0)) = U_0^1 V_0^{2m} + \ldots + U_0^{2m} V_0^1. \tag{11}$$

The right-hand side of (9) assumes k complex values. The expression $\lambda = \lambda_0 + \varepsilon^{\frac{1}{k}}\lambda_1 + o(\varepsilon^{\frac{1}{k}})$ describes the splitting of a k-fold eigenvalue as the parameters are varied along a curve issuing in the direction $\mathbf{e}$ if the radicand in (9) is nonzero. Specifically, for $k = 1$, relations (6) and (9) describe the behavior of a simple eigenvalue.

2. NONCONSERVATIVE STABILITY PROBLEMS

As an example, we consider a uniform elastic cantilevered column (Fig. 1). It is assumed that the free end of the column is loaded by a nonconservative force q, whose direction is determined by the parameter $\eta \in [0, 1]$. The case $\eta = 1$ corresponds to the column loaded by a tangential follower force (the Beck problem [10, 11]). If $\eta = 0$, then q is a potential (conservative) force. The dimensionless differential equation describing small-amplitude vibrations of the column in the Oxy plane has the form [9]

$$y''''(x, \tau) + q y''(x, \tau) + \ddot{y}(x, \tau) = 0,$$

where the dotted and primed variables denote derivatives with respect to time τ and x, respectively. Separating the variables $[y(x, \tau) = u(x)e^{i\tau\sqrt{\lambda}}]$, we arrive at the eigenvalue problem [9]

$$l(u) \equiv u'''' + q u'' = \lambda u, \tag{12}$$

$$U^1(u) \equiv u(0) = 0, \quad U^2(u) \equiv u'(0) = 0,$$
$$U^3(u) \equiv u''(1) = 0, \tag{13}$$
$$U^4(u) \equiv u'''(1) + (1 - \eta) q u'(1) = 0.$$

Its adjoint problem is

$$l^*(v) \equiv v'''' + q v'' = \lambda v, \tag{14}$$

$$V^1(v) \equiv -v(0) = 0, \quad V^2(v) \equiv v'(0) = 0,$$
$$V^3(v) \equiv v''(1) + \eta q v(1) = 0, \tag{15}$$
$$V^4(v) \equiv -v'''(1) - q v'(1) = 0,$$

Using (11), we find for V^5, V^6, V^7, and V^8 that

$$V^5 \equiv v(1), \quad V^6 \equiv -v'(1), \quad V^7 \equiv -v''(0) - qv(0),$$
$$V^8 \equiv v'''(0) + qv'(0). \tag{16}$$

The general solution to Eq. (12) is written as

$$u(x) = C_1 \cosh(ax) + C_2 \sinh(ax)$$
$$+ C_3 \cos(bx) + C_4 \sin(bx), \tag{17}$$

where $a = \left(-\frac{q}{2} + \left(\frac{q^2}{4} + \lambda\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$ and $b = \left(\frac{q}{2} + \left(\frac{q^2}{4} + \lambda\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}$. The substitution of (17) into (13) gives the existence condition for a nontrivial solution $u(x)$ to the eigenvalue problem [9]:

$$D(\lambda, \eta, q) \equiv (2\lambda + (1-\eta)q^2)(1 + \cosh(a)\cos(b))$$
$$+ q(2\eta - 1)(q + ab\sinh(a)\sin(b)) = 0. \tag{18}$$

The eigenvalues $\lambda(\eta, q)$ are determined by this equation. The linear nonconservative mechanical system under consideration is the so-called circulatory system. It is stable if and only if all eigenvalues λ are positive and semisimple. If all $\lambda \in \mathbb{R}$ are real and some of them are negative, then the circulatory system is statically unstable (divergence). The existence of at least one $\lambda \in \mathbb{C}$ means dynamic instability (flutter) [10, 11].

The characteristic determinant $D(\lambda, \mathbf{p})$ is a smooth function of the spectral parameter λ and the vector $\mathbf{p} = (\eta, q)$. For every fixed $\mathbf{p} = \mathbf{p}_0$, the spectrum of the operator L defined by (12) and (13) is discrete [4]. Its eigenvalues may be simple or multiple roots of $D(\lambda, \mathbf{p}_0)$. Suppose that the equation $D(\lambda, \mathbf{p}_0) = 0$ has a k-fold real root $\lambda = \lambda_0$ at $\mathbf{p} = \mathbf{p}_0$. Then, by the Malgrange preparation theorem [12], there exists a neighborhood of the point $(\lambda_0, \mathbf{p}_0)$ in which $D(\lambda, \mathbf{p})$ is given by

$$D(\lambda, \mathbf{p}) = \left[(\lambda - \lambda_0)^k + \sum_{i=0}^{k-1} (\lambda - \lambda_0)^i a_i(\mathbf{p})\right] b(\lambda, \mathbf{p}), \tag{19}$$

where $a_0(\mathbf{p}), \ldots, a_{k-1}(\mathbf{p})$ and $b(\lambda, \mathbf{p})$ are smooth functions such that $a_i(\mathbf{p}_0) = 0$ and $b(\lambda_0, \mathbf{p}_0) \neq 0$.

For example, let λ_0 be a simple real root of $D(\lambda, \mathbf{p}_0) = 0$. Then, under small variations in the parameters in the neighborhood of $\mathbf{p}_0$, it remains real and simple, because $\lambda = \lambda_0 - a_0(\mathbf{p})$ by virtue of (19). Therefore, if all eigenvalues of L are positive and simple at $\mathbf{p} = \mathbf{p}_0$, then $\mathbf{p}_0$ is an interior point of the stability region of the circulatory system (12), (13). In a similar fashion, it is established that the points of the parameter plane that correspond to operators whose spectrum contains a simple zero or double real eigenvalue with a Keldysh chain of length 2 constitute smooth curves. The stability of the nonconservative system in the neighborhood of such a curve depends on the behavior of the zero or double eigenvalue with varying parameters. In the first approximation, this behavior is described by relations (6) and (9), where we set $k = 1$ or 2:

$$\lambda = \varepsilon\langle \mathbf{f}_1, \mathbf{e}\rangle + o(\varepsilon), \tag{20}$$

$$\lambda = \lambda_0 \pm \sqrt{\varepsilon\langle \mathbf{f}_2, \mathbf{e}\rangle} + o\left(\varepsilon^{\frac{1}{2}}\right). \tag{21}$$

Formulas (20) and (21) show that bifurcations of the eigenvalues are determined by the signs of the scalar products $\langle \mathbf{f}_1, \mathbf{e}\rangle$ and $\langle \mathbf{f}_2, \mathbf{e}\rangle$. The corresponding inequalities are linear approximations of the stability, flutter, and divergence regions, and the vectors $\mathbf{f}_1$ and $\mathbf{f}_2$ are the normals to the boundary of the divergence region determined by the simple zero eigenvalues and to the boundary of the flutter region determined by the double real eigenvalues with a Keldysh chain of length 2.

The possible values of the parameters η and q at which system (12), (13) becomes statically instable are found from Eq. (18), where we set $\lambda = 0$:

$$\cos(\sqrt{q}) = \frac{\eta}{\eta - 1}. \tag{22}$$

Equation (22) defines the curve of zero eigenvalues on the parameter plane (η, q). A fragment of this curve separates the stability and divergence regions. The curve of double eigenvalues is determined by numerical evaluation of the roots of the characteristic equation (18) for various values of η and q. The curves found divide the parameter plane (η, q) into the stability (S), flutter (F), and divergence (D) regions (Fig. 2).

Figure 2 shows that the flutter region is contiguous with the divergence and stability regions. Consequently, in motion along the boundary of the flutter region, the double eigenvalue changes its sign at some point. Let us find it. According to (4), the orthogonality condition $\int_0^1 u_0 v_0 \, dx = 0$ must be satisfied at the boundary points of the flutter region. The eigenfunctions u_0 and v_0 of the zero eigenvalue are defined by the formulas

$$u_0 = \sin(b) - xb\cos(b)$$
$$- \sin(b)\cos(bx) + \cos(b)\sin(bx), \tag{23}$$

$$v_0 = 1 - \cos(bx), \quad b = \sqrt{q_0}. \tag{24}$$

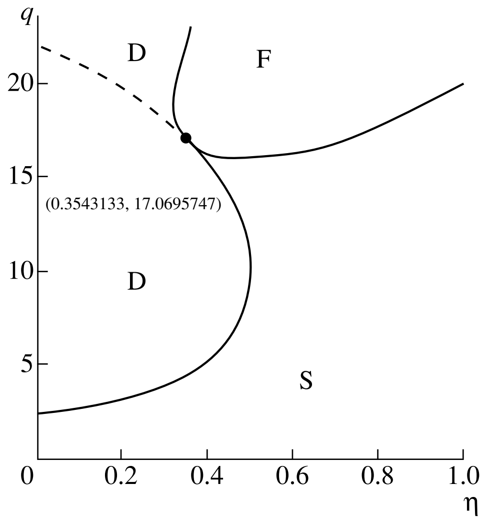

Fig. 2. Stability diagram.

The evaluation of the integral gives an equation satisfied by the ordinate of the desired point:

$$q_0 = (\sqrt{q_0} - 2\sin(\sqrt{q_0}))$$
$$\times(\sqrt{q_0}(1 + 2\cos(\sqrt{q_0})) - 4\sin(\sqrt{q_0})). \qquad (25)$$

For $q_0 > 0$, the minimum of the solution set of Eq. (25) is $q_0 = 17.0695748$. Substituting this into the curve equation (22), we find the corresponding value of the second parameter $\eta_0 = 0.35431330$.

Thus, at the point $\mathbf{p}_0 = (0.35431330, 17.0695748)$, there exists a double zero eigenvalue λ_0 with a Keldysh chain of length 2. The bifurcation of such an eigenvalue is given by (21). In (10), we substitute (12) for $l(u)$ and (13) and (16) for U^1, U^2, U^3, U^4, V^5, V^6, V^7, and V^8 and use conditions (4) and (5) to obtain an expression for the normal to the boundary:

$$\mathbf{f}_2 = \left(\frac{q_0 u_0'(1) v_0(1)}{\int_0^1 u_0 v_1 dx}, \frac{\int_0^1 u_0'' v_0 dx - (1-\eta_0) u_0'(1) v_0(1)}{\int_0^1 u_0 v_1 dx} \right). \qquad (26)$$

In addition to the eigenfunctions u_0 and v_0 corresponding to the double zero eigenvalue, we need the associated function v_1 to calculate the vector $\mathbf{f}_2$. At $k = 2$ and $\lambda_0 = 0$, we solve the boundary value problem (3) supplemented with the differential expression and the boundary conditions from (14) and (15) to obtain

$$v_1 = \frac{x + x^2}{2b^2} + \frac{x-1}{2b^3}\sin(bx)$$
$$+ \frac{b^2\cos(b) - \sin^2(b)}{b^4(b\cos(b) - \sin(b))}(\sin(bx) - bx), \qquad (27)$$

where $b = \sqrt{q_0}$. The possibility of appearance of associated functions in the Beck problem was noted in [13], but explicit expressions for them were not found.

Substituting eigenfunctions (23) and (24) and the associated function (27) into (26), we find the normal to the boundary of the flutter region at the point $\mathbf{p}_0$:

$$\mathbf{f}_2 = (-24\,288.8139, -1024.49949).$$

Given the normal vector, we can inspect the neighborhood of the boundary point of the flutter region in all directions $\mathbf{e}$ such that $\langle \mathbf{f}_2, \mathbf{e} \rangle \neq 0$. Specifically, for two orthogonal directions $\mathbf{e} = (1, 0)$ and $\mathbf{e} = (0, 1)$, we obtain

$$\lambda = \pm 155.848689\sqrt{\eta_0 - \eta},$$
$$\lambda = \pm 32.0078037\sqrt{q_0 - q}, \qquad (28)$$

respectively. Typically, in splitting a double zero, either a complex conjugate pair or two real eigenvalues, one of which is negative, are formed (Table 1). Therefore,

Table 1. Splitting of the double eigenvalue $\lambda_0 = 0$ in the neighborhood of the singular point $\mathbf{p}_0 = (0.35431330, 17.0695748)$

e	$\eta - \eta_0$	$q - q_0$	λ (28)	λ (18)
(0, 1)	0	10^{-4}	$\mathrm{Re}\lambda_{1,2} = 0$	$\mathrm{Re}\lambda_{1,2} = -0.00151188$
			$\mathrm{Im}\lambda_{1,2} = \pm 0.32007804$	$\mathrm{Im}\lambda_{1,2} = \pm 0.32007586$
(0, –1)	0	10^{-4}	$\lambda_1 = 0.32007804$	$\lambda_1 = 0.32159210$
			$\lambda_2 = -0.32007804$	$\lambda_2 = -0.31856833$
(1, 0)	10^{-4}	0	$\mathrm{Re}\lambda_{1,2} = 0$	$\mathrm{Re}\lambda_{1,2} = 0.02668744$
			$\mathrm{Im}\lambda_{1,2} = \pm 1.55848689$	$\mathrm{Im}\lambda_{1,2} = \pm 1.55823291$
(–1, 0)	-10^{-4}	0	$\lambda_1 = 1.55848689$	$\lambda_1 = 1.53205170$
			$\lambda_2 = -1.55848689$	$\lambda_2 = -1.58543004$

the normal vector $\mathbf{f}_2$ at $\mathbf{p}_0$ is directed toward the divergence region, the inequality $\langle \mathbf{f}_2, \mathbf{e} \rangle > 0$ defines a tangent cone to the divergence region, and $\langle \mathbf{f}_2, \mathbf{e} \rangle < 0$ defines a tangent cone to the flutter region (see Fig. 2). Only curves issued along the tangent to the boundary can lead from the singular point toward the stability region. The direction of the corresponding tangent vector $\mathbf{e}_*$ can be found by considering the bifurcations of the double zero in the degenerate case $\langle \mathbf{f}_2, \mathbf{e}_* \rangle = 0$.

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