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# BIFURCATION OF EIGENVALUES OF NONSELFADJOINT DIFFERENTIAL OPERATORS IN NONCONSERVATIVESTABILITY PROBLEMS

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## ABSTRACT

In the present paper eigenvalue problems for non-selfadjoint linear differential operators smoothly dependent on a vector of real parameters are considered. Bifurcation of eigenvalues along smooth curves in the parameter space is studied. The case of multiple eigenvalue with Keldysh chain of arbitrary length is considered. Explicit expressions describing bifurcation of eigenvalues are found. The obtained formulae use eigenfunctions and associated functions of the adjoint eigenvalue problems as well as the derivatives of the differential operator taken at the initial point of the parameter space. These results are important for the stability theory, sensitivity analysis and structural optimization. As a mechanical application the extended Beck's problem of stability of an elastic column under action of potential force and tangential follower force is considered and discussed in detail.

## NOMENCLATURE

$A$ The cross-sectional area of the column.
$E$ The Young modulus of the column.
$I$ The cross-sectional moment of inertia of the column.
$EI$ The bending stiffness of the column.
$L_c$ The length of the column.
$\langle \mathbf{a}, \mathbf{b} \rangle$ The scalar product $\sum_{i=1}^{n} a_i b_i$ of vectors $\mathbf{a}, \mathbf{b} \in R^n$ in the parameter space.
$(\varphi, \psi)$ The scalar product $\int_0^1 \varphi(x)\overline{\psi(x)}dx$ of functions $\varphi$, $\psi \in C^{(m)}\ [0,1]$.
$\rho$ The material density of the column.
$\rho A$ The mass of the column per unit length.

## INTRODUCTION

Non-selfadjoint operators appear in nonconservative problems of mechanics and physics. The theory of non-selfadjoint operators ascending to works by G. Birkhoff was then developing by many mathematicians. M.V. Keldysh was the first who generalized the notion of the Jordan chain of vectors for a wide class of non-selfadjoint operators (Keldysh, 1951). In the generic case the spectrum of a multiparameter family of non-selfadjoint operators contains multiple eigenvalues with Keldysh chains. It turns out that such eigenvalues define geometric properties of the stability boundary of a corresponding non-conservative system. An effective tool of analysis of this boundary is the study of a bifurcation of eigenvalues due to change of parameters. Up to the recent time stability boundaries only of finite-dimensional systems were investigated (Seyranian, 1991).

Consider an eigenvalue problem for the linear differential operator $L$

$$l(u) = \lambda u, \quad U^s(u) = 0, \quad s = 1, \ldots, m, \tag{1}$$

where

$$l(u) \equiv \sum_{i=0}^{m} a_i \frac{d^{m-i}u}{dx^{m-i}},$$

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$$U^s(u) \equiv \sum_{i=0}^{m-1} \left( \alpha_i^s \frac{d^i u}{dx^i}\bigg|_{x=0} + \beta_i^s \frac{d^i u}{dx^i}\bigg|_{x=1} \right).$$

Operators $U^s(u)$ are linear forms with respect to the variables $u(0)$, $u'(0)$, $\dots$, $u^{(m-1)}(0)$; $u(1)$, $u'(1)$, $\dots$, $u^{(m-1)}(1)$. These variables are values of the function $u \in C^{(m)}[0,1]$ and the derivatives of this function up to $(m-1)$ – th order taken at the points $x = 0$ and $x = 1$. It is assumed that forms $U^s$, $s = 1, 2, \dots, m$ are linearly independent.

The differential expression

$$l^*(v) \equiv \sum_{i=0}^{m} (-1)^{m-i} \overline{a_i} v^{(m-i)},$$

where the overbar denotes complex conjugation, is called *adjoint* to the differential expression $l(u)$ (Naimark, 1969). With the use of integration by parts it can be shown that

$$\int_0^1 l(u)\bar{v}dx = P(\alpha, \beta) + \int_0^1 u\overline{l^*(v)}dx, \tag{2}$$

where $P(\alpha, \beta)$ – is a bilinear form of variables

$$\alpha = (u(0), u'(0), \dots, u^{(m-1)}(0), u(1), u'(1), \dots, u^{(m-1)}(1)) \tag{3}$$

$$\beta = (v(0), v'(0), \dots, v^{(m-1)}(0), v(1), v'(1), \dots, v^{(m-1)}(1)) \tag{4}$$

Let us choose the forms $U^{m+1}, U^{m+2}, \dots, U^{2m}$ so that $U^1, U^2, \dots, U^{2m}$ be linearly independent. Then variables (3), (4) can be expressed as linear combinations of the forms $U^1, U^2, \dots, U^{2m}$. Substituting these linear combinations into (2), we get Lagrange's formula (Naimark, 1969)

$$(l(u), v) - (u, l^*(v)) = U^1 V^{2m} + \dots + U^{2m} V^1. \tag{5}$$

The coefficients at $U^1, U^2, \dots, U^{2m}$ are linear forms with respect to variables (3), (4) and are denoted by $V^{2m}, \dots, V^2, V^1$, respectively. The forms $V^1, V^2, \dots, V^{2m}$ are linearly independent. The boundary conditions $V^s(v) = 0$, $s = 1, 2, \dots, m$ are called adjoint to boundary conditions (1). The differential operator $L^*$, corresponding to the differential expression $l^*(v)$ and to the adjoint boundary conditions, is called adjoint to the operator $L$, and we say that the eigenvalue problem

$$l^*(v) = \overline{\lambda} v, \quad V^s(v) = 0, \quad s = 1, \dots, m, \tag{6}$$

is adjoint to eigenvalue problem (1).

Due to boundary conditions (1), (6) formula (5) for the adjoint operators $L$ and $L^*$ takes a simple form: $(l(u), v) = (u, l^*(v))$. If we consider differential expression $l(u)$ and assume that the function $u$ satisfies the non-homogeneous boundary conditions

$$U^s(u) = G^s, \quad s = 1, 2, \dots, m, \tag{7}$$

then Lagrange's formula (5) takes the form

$$(l(u), v) - (u, l^*(v)) = G^1 V^{2m} + \dots + G^m V^{m+1}. \tag{8}$$

This is valid since $v$ satisfies boundary conditions (6).

## COLLAPSE OF KELDYSH CHAINS

Suppose that in eigenvalue problem (1) the coefficients of the differential expression $l(u)$ and the coefficients of the forms $U^s(u)$ are real functions, *smoothly* dependent on a vector of real parameters $\mathbf{p} = (p_1, p_2, \dots, p_n)$, i.e. are $C^\infty$ – functions on an open set $\Omega \subset R^n$. Let $\lambda_0$ be an eigenvalue of the operator $L$ at the point $\mathbf{p} = \mathbf{p}_0$. We are interested in bifurcation of eigenvalues along the curves $\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon \mathbf{e} + o(\epsilon)$, emitted from the initial point $\mathbf{p}_0$ in the parameter space. The vector $\mathbf{e} = (e_1, e_2, \dots, e_n)$ defines the direction of a curve and $\epsilon \geq 0$ is a small parameter. Due to variation of parameters the differential expression $l(u)$ and the forms $U^s(u)$ take increments

$$l(u) = l_0(u) + \epsilon l_1(u) + \dots, \quad U^s(u) = U_0^s(u) + \epsilon U_1^s(u) + \dots, \tag{9}$$

Differential expressions $l_0(u)$, $l_1(u)$ look like

$$l_0 = l(u)|_{\mathbf{p}=\mathbf{p}_0}, \quad l_1(u) = \sum_{i=1}^{n} e_i \frac{\partial l}{\partial p_i}(u), \tag{10}$$

and for the forms $U_0^s(u)$, $U_1^s(u)$ we have

$$U_0^s = U^s(u)|_{\mathbf{p}=\mathbf{p}_0}, \quad U_1^s(u) = \sum_{i=1}^{n} e_i \frac{\partial U^s}{\partial p_i}(u). \tag{11}$$

All the derivatives in formulae (10), (11) are taken at the point $\mathbf{p} = \mathbf{p}_0$. Thus, we consider *regular* perturbations which do not increase the order of the non-perturbed operator $L_0 = L(\mathbf{p}_0)$ (Vishik and Lyusternik, 1960).

Consider an eigenvalue $\lambda_0$ with *a Keldysh chain of length* $k$. This means that at $\mathbf{p} = \mathbf{p}_0$ there exist an eigenfunction $u_0(x)$ and associated functions $u_1(x)$, $u_2(x)$, ..., $u_{k-1}(x)$, corresponding to the $\lambda_0$ and satisfying the equations and the boundary conditions

$$\begin{array}{ll} l_0(u_0) = \lambda_0 u_0, & U_0^s(u_0) = 0; \\ l_0(u_i) = \lambda_0 u_i + u_{i-1}, & U_0^s(u_i) = 0; \\ i = 1, \ldots, k-1; & s = 1, \ldots, m. \end{array} \tag{12}$$

The adjoint eigenvalue problem looks like

$$\begin{array}{ll} l_0^*(v_0) = \overline{\lambda_0} v_0, & V_0^s(v_0) = 0; \\ l_0^*(v_i) = \overline{\lambda_0} v_i + v_{i-1}, & V_0^s(v_i) = 0; \\ i = 1, \ldots, k-1; & s = 1, \ldots, m. \end{array} \tag{13}$$

The notion of Keldysh chain is an analogue of Jordan chain of vectors in eigenvalue problems for differential operators (Keldysh, 1951; Naimark, 1969; Gohberg & al., 1982). Eigenfunctions and associated functions of adjoint eigenvalue problems (12), (13) are related by the following conditions

$$\begin{array}{c} (u_j, v_0) = 0, \quad j = 0, \ldots, k-2, \\ (u_{k-1}, v_0) \equiv (u_0, v_{k-1}) \neq 0; \end{array} \tag{14}$$

$$(u_{j-1}, v_i) \equiv (u_j, v_{i-1}), \quad i, j, = 1, \ldots, k-1 \tag{15}$$

that can be proved by equations (12) and (13) with the use of the relation $(l(u), v) = (u, l^*(v))$ stated for the adjoint operators.

Taking a variation of the vector of parameters $\mathbf{p} = \mathbf{p}_0 + \epsilon \mathbf{e} + o(\epsilon)$ leads to the perturbation of eigenvalues and eigenfunctions. In the case of a multiple eigenvalue with the Keldysh chain of length $k$ the expansions of eigenvalues and eigenfunctions contain terms with fractional powers of the small parameter $\epsilon^{j/k}$, $j = 0, 1, 2, \ldots$ (Vishik and Lyusternik, 1960):

$$\begin{array}{c} \lambda = \lambda_0 + \epsilon^{1/k}\lambda_1 + \epsilon^{2/k}\lambda_2 + \epsilon^{3/k}\lambda_3 + \ldots \\ u = u_0 + \epsilon^{1/k}w_1 + \epsilon^{2/k}w_2 + \epsilon^{3/k}w_3 + \ldots \end{array} \tag{16}$$

Substituting expansions (9) and (16) into (1), we get expressions which determine the first order perturbations of the eigenvalue $\lambda_0$ and the eigenfunction $u_0$

$$\begin{array}{c} l_0(w_1) - \lambda_0 w_1 = \lambda_1 u_0, \quad U_0^s(w_1) = 0, \\ l_0(w_2) - \lambda_0 w_2 = \lambda_2 u_0 + \lambda_1 w_1, \quad U_0^s(w_2) = 0, \\ \ldots \\ l_0(w_{k-1}) - \lambda_0 w_{k-1} = \\ \lambda_{k-1} u_0 + \lambda_{k-2} w_1 + \ldots + \lambda_1 w_{k-2}, \quad U_0^s(w_{k-1}) = 0; \end{array} \tag{17}$$

$$\begin{array}{c} l_0(w_k) - \lambda_0 w_k = \\ \lambda_k u_0 + \lambda_{k-1} w_1 + \ldots + \lambda_1 w_{k-1} - l_1(u_0), \quad U_0^s(w_k) = -U_1^s(u_0). \end{array} \tag{18}$$

Functions $w_j$ can be found from equations (12) and (17) in the form

$$w_j = \lambda_1^j u_j + \sum_{p=0}^{j-1} \gamma_{jp} u_p, \quad j = 1, \ldots, k-1, \tag{19}$$

where $\gamma_{jp}$ are arbitrary constants. Consider the inner product of the function $v_0$ with the left and right hand sides of (18). Using then expression (19) for $w_j$, equations (14), (15), and Lagrange's formula (8), which in this case has the form

$$(l_0(w_k) - \lambda_0 w_k, v_0) - (w_k, l_0^*(v_0) - \overline{\lambda_0} v_0) = -\sum_{s=1}^{m} U_1^s(u_0) V_0^{2m-s+1}(v_0),$$

we get the coefficient $\lambda_1$ in the expansion of the eigenvalue $\lambda$

$$\lambda_1^k = \frac{(l_1(u_0), v_0) - \sum_{s=1}^m U_1^s(u_0) V_0^{2m-s+1}(v_0)}{(u_{k-1}, v_0)}. \tag{20}$$

With the use of equations (10) and (11) we can write expression (20) in the form (Seyranian, 1991)

$$\lambda_1 = \sqrt[k]{\langle \mathbf{f}_k, \mathbf{e} \rangle + i \langle \mathbf{g}_k, \mathbf{e} \rangle}, \tag{21}$$

where the real vectors $\mathbf{f}_k$ and $\mathbf{g}_k$ correspond to the $k$ – fold eigenvalue $\lambda_0$ at the point $\mathbf{p} = \mathbf{p}_0$ and their components are defined by

$$f_k^j + i g_k^j = \frac{(\frac{\partial l}{\partial p_j}(u_0), v_0) - \sum_{s=1}^m \frac{\partial U^s}{\partial p_j}(u_0) V_0^{2m-s+1}(v_0)}{(u_{k-1}, v_0)}, \tag{22}$$

The right hand side of (21) takes $k$ complex values. The expression $\lambda = \lambda_0 + \epsilon^{1/k}\lambda_1 + o(\epsilon^{1/k})$ describes the splitting of the $k$ – fold eigenvalue due to change of parameters along a curve emitted in direction $\mathbf{e}$, if the radicand in (21) is not zero. In particular, when $k = 1$ equations (16), (21) describe the behavior of a simple eigenvalue, and when $k = 2$ – the splitting of a double eigenvalue $\lambda_0$ with the Keldysh chain of length 2.

## NON-CONSERVATIVE STABILITY PROBLEMS

As an example of continuous non-conservative mechanical system we consider a uniform elastic cantilever column, Fig. 1. We assume that the non-conservative force $P$, which can be represented as the sum of a tangential follower force and a potential load, is acting at the free end of the column. Parameter $\eta \in [0,1]$ measures the non-conservativity of the force $P$. The case $\eta = 1$ means that the column is loaded by purely tangential follower force (Beck's problem). If $\eta = 0$, then the force $P$ is potential (conservative). Let us introduce the non-dimensional variables: the coordinate $x$, the deflection $y$, the time $\tau$, and the force $q$

$$x=X/L_c, \quad y=Y/L_c, \quad \tau=t/\sqrt{\rho A L_c^4/EI}, \quad q=PL_c^2/EI.$$

We consider the transverse vibrations of the column in the plane $Oxy$, Fig. 1. The differential equation describing small in-plane vibrations of the column has the form (Pedersen, 1977)

$$y''''(x,\tau) + qy''(x,\tau) + \ddot{y}(x,\tau) = 0.$$

Dots mean differentiation with respect to time $\tau$ and primes denote differentiation w.r.t. coordinate $x$. Separating time by $y(x,\tau) = u(x)exp\left(i\sqrt{\lambda}\tau\right)$, we get the eigenvalue problem (Pedersen, 1977)

$$l(u) = u'''' + qu'' = \lambda u, \tag{23}$$

$$\begin{aligned} &U^1(u) \equiv u(0) = 0, && U^3(u) \equiv u''(1) = 0,\\ &U^2(u) \equiv u'(0) = 0, && U^4(u) \equiv u'''(1) + (1-\eta)qu'(1) = 0. \end{aligned} \tag{24}$$

The corresponding adjoint eigenvalue problem looks like

$$l^*(v) \equiv v'''' + qv'' = \lambda v, \tag{25}$$

$$\begin{aligned} &V^1(v) \equiv -v(0) = 0, \quad V^3(v) \equiv v''(1) + \eta q v(1) = 0,\\ &V^2(v) \equiv v'(0) = 0, \quad V^4(v) \equiv -v'''(1) - qv'(1) = 0, \end{aligned} \tag{26}$$

and for the forms $V^5 \dots V^8$ we have

$$\begin{aligned} &V^5 \equiv v(1), && V^7 \equiv -v''(0) - qv(0),\\ &V^6 \equiv -v'(1), && V^8 \equiv v'''(0) + qv'(0). \end{aligned} \tag{27}$$

General solution of equation (23) is

$$u(x) = C_1\cosh(ax) + C_2\sinh(ax) + C_3\cos(bx) + C_4\sin(bx).$$

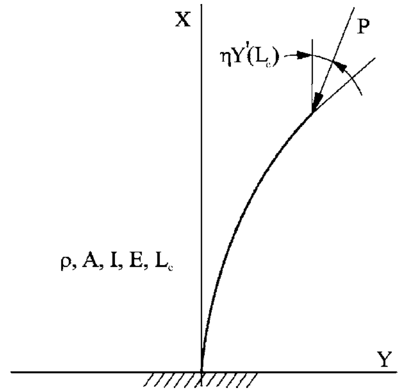


Figure 1. THE COLUMN LOADED BY A NONCONSERVATIVE FORCE.

Substituting the general solution into boundary conditions (24) we obtain the condition of existing of a non-trivial solution $u(x)$ to eigenvalue problem (23), (24)

$$\begin{aligned} D(\lambda,\eta,q) \equiv (2\lambda + (1-\eta)q^2)(1+\cosh(a)\cos(b)) + \\ +q(2\eta-1)(q+ab\sinh(a)\sin(b)) = 0, \end{aligned} \tag{28}$$

(Pedersen, 1977), where

$$a = \sqrt{-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \lambda}}, \quad b = \sqrt{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \lambda}}.$$

Equation (28) gives eigenvalues $\lambda$, depending on parameters $\eta$ and $q$.

The two-parameter mechanical system under consideration belongs to the so called *circulatory systems*. It is known that a circulatory system is stable if and only if all the eigenvalues $\lambda$ are positive and semisimple. If all $\lambda$ are real and some of them negative then the circulatory system is statically unstable (divergence). Existence of at least one complex eigenvalue means flutter instability (Bolotin, 1963; Ziegler, 1965).

Characteristic determinant $D(\lambda, \mathbf{p})$ is a smooth function of the spectral parameter $\lambda \in R$ and the vector $\mathbf{p} = (\eta, q)$. At any fixed value $\mathbf{p} = \mathbf{p}_0$ the spectrum of the operator $L$, which is defined by formulae (23) and (24), is discrete (Naimark, 1969). The eigenvalues of the operator can be simple or multiple roots of the function $D(\lambda, \mathbf{p}_0)$. If at $\mathbf{p} = \mathbf{p}_0$ the equation $D(\lambda, \mathbf{p}_0) = 0$ has the $k$ – fold real root $\lambda = \lambda_0$, then according to Malgrange's preparation theorem (Chow and Hale, 1982) there exists a neighborhood $U_0 \subset R \times R^n$ of the point $(\lambda_0, \mathbf{p}_0)$, where $D(\lambda, \mathbf{p})$ has the form

$$D(\lambda, \mathbf{p}) = \left[(\lambda - \lambda_0)^k + \sum_{i=0}^{k-1} (\lambda - \lambda_0)^i a_i(\mathbf{p})\right] b(\lambda, \mathbf{p}). \quad (29)$$

The functions $a_0(\mathbf{p})$, ..., $a_{k-1}(\mathbf{p})$; $b(\lambda, \mathbf{p})$ are smooth, and $a_i(\mathbf{p}_0) = 0$, $b(\lambda_0, \mathbf{p}_0) \neq 0$.

Let for example $\lambda_0$ be a simple real root of the equation $D(\lambda_0, \mathbf{p}_0) = 0$. Then, due to (29) we can write $\lambda = \lambda_0 - a_0(\mathbf{p})$, and $\lambda_0$ remains real and simple in some neighborhood of the point $\mathbf{p}_0$. Thus, if at $\mathbf{p} = \mathbf{p}_0$ all the eigenvalues of the operator $L$ are positive and simple, then $\mathbf{p}_0$ is the inner point of the stability domain of circulatory system (23),(24). Similarly, the points of the parameter plane, corresponding to the operators, which spectra contain either simple zero eigenvalue or real double eigenvalue with Keldysh chain of length 2, form smooth curves. It is clear that the stability of the system in the vicinity of such curves depends on behavior of the zero or the double eigenvalue due to change of parameters. According to (16), (21), where we should put $k = 1$ or $k = 2$, the behavior of the simple zero eigenvalue is described by the formula

$$\lambda = \epsilon \langle \mathbf{f}_1, \mathbf{e} \rangle + o(\epsilon), \quad (30)$$

and the splitting of the real double $\lambda_0$ is governed by the expression

$$\lambda = \lambda_0 \pm \sqrt{\epsilon \langle \mathbf{f}_2, \mathbf{e} \rangle} + o(\epsilon^{1/2}). \quad (31)$$

The inequality $\langle \mathbf{f}_1, \mathbf{e} \rangle > 0$ defines a set of such directions $\mathbf{e}$ that the curves $\mathbf{p} = \mathbf{p}(\epsilon)$ emitted along these vectors lie in the stability domain, i.e. *a tangent cone* to the stability domain. The eigenvalue $\lambda_0 = 0$ becomes negative at $\langle \mathbf{f}_1, \mathbf{e} \rangle < 0$. Consequently, this inequality gives a tangent cone to the static instability (divergence) domain. The eigenvalue remains zero up to the terms of order $\epsilon^2$ on the curves, emitted in the directions $\mathbf{e}$, such that $\langle \mathbf{f}_1, \mathbf{e} \rangle = 0$. Thus, the equation $\langle \mathbf{f}_1, \mathbf{p} - \mathbf{p}_0 \rangle = 0$ defines a tangent line to the curve, where the operator $L$ has simple zero eigenvalue. If other eigenvalues remain simple and positive along this curve, then it forms a boundary between stability and divergence domains. The vector $\mathbf{f}_1$ is a normal vector to the boundary and is directed to the stability domain. Analyzing splitting of the double eigenvalue with the formula (31) we can show that the points of the parameter plane, corresponding to the operators with the real double eigenvalue with Keldysh chain of length 2, belong to the smooth

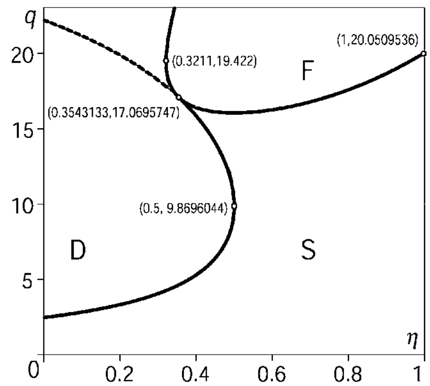


Figure 2. STABILITY DIAGRAM.

parts of the boundary between the flutter domain and the stability domain if $\lambda_0 > 0$ or divergence domain if $\lambda_0 < 0$. In this case the vector $\mathbf{f}_2$ is normal to the boundary.

Possible values of parameters $\eta$ and $q$ at which the system loses stability statically follows from equation (28) where we put $\lambda = 0$

$$\cos(\sqrt{q}) = \frac{\eta}{\eta - 1}. \quad (32)$$

Equation (32) defines the curve of simple zero eigenvalues, the part of which forms the boundary between stability and divergence domains on the plane of parameters $(\eta, q)$. Calculating the roots of characteristic equation (28) at different values $q$ (when the parameter $\eta$ is fixed) we find approximately the point, where two simple eigenvalues merge into a double real eigenvalue. To obtain the flutter boundary it is necessary to solve this problem at different values of parameter $\eta$. The curves found subdivide the plane of parameters $(\eta, q)$ into stability (S), flutter (F), and divergence (D) domains, Fig. 2.

It can be seen from Fig. 2 that flutter domain has a common boundary with stability and divergence domains. Thus, the double eigenvalue changes its sign at some point of the flutter boundary. Remind that in accordance with (14) the orthogonality condition $\int_0^1 u_0 \overline{v_0} dx = 0$ must be true at the points of the flutter boundary. This fact allows us to find the point, corresponding to a double zero eigenvalue. The eigenfunctions $u_0$ and $v_0$ of the zero eigenvalue can be chosen real-valued and are defined by the formulae

$$u_0 = \sin(b) - xb\cos(b) - \sin(b)\cos(bx) + \cos(b)\sin(bx), \quad (33)$$

$$v_0 = 1 - \cos(bx), \quad b = \sqrt{q_0}. \tag{34}$$

The functions $u_0$ and $v_0$ are solutions of eigenvalue problems (23) – (26) at $\lambda = 0$. After integration we come to the transcendental equation, which the ordinate of the point under study must satisfy

$$q_0 = (\sqrt{q_0} - 2\sin(\sqrt{q_0}))(\sqrt{q_0}(1 + 2\cos(\sqrt{q_0})) - 4\sin(\sqrt{q_0})). \tag{35}$$

The minimal element of the set of solutions of equation (35) at $q_0 > 0$ is $q_0 = 17.0695748$. Substituting this solution into equation (32), we find the corresponding value of the second parameter $\eta_0 = 0.35431330$.

Thus, at the point $\mathbf{p}_0 = (0.35431330, 17.0695748)$ there exists the double eigenvalue $\lambda_0 = 0$ with Keldysh chain of length 2. The bifurcation of such eigenvalue is described by formula (31). Substituting the differential expression $l(u)$ from (23), the forms $U^1, \ldots, U^4$ and $V^5, \ldots, V^8$ from (24), (27) into formula (22), and using conditions (14), we get the expressions for the normal vector to the boundary

$$\mathbf{f}_2 = \left( \frac{q_0 u_0'(1) v_0(1)}{\int_0^1 u_0 \overline{v_1} dx}, \frac{\int_0^1 u_0'' v_0 dx - (1-\eta_0) u_0'(1) v_0(1)}{\int_0^1 u_0 \overline{v_1} dx} \right). \tag{36}$$

To compute the vector $\mathbf{f}_2$ it is necessary to know the associated functions $u_1$, $v_1$ as well as the eigenfunctions $u_0, v_0$, corresponding to the double zero eigenvalue. Solving at $k = 2$ and $\lambda = 0$ boundary value problems (12), (13) with differential expressions and boundary conditions from (23)–(26) we get the real-valued functions

$$u_1 = -\frac{\cot(b)}{6b} x^3 + \frac{1}{2b^2} x^2 + \frac{\cot(b)(\cos(bx) - 1) + \sin(bx)}{2b^3} x + \frac{(bx - \sin(bx))(b + 2b\cos(b) - 2\sin(b))}{2b^4 \sin^2(b)}, \tag{37}$$

$$v_1 = \frac{x + x^2}{2b^2} + \frac{x-1}{2b^3} \sin(bx) + \frac{b^2 \cos(b) - \sin^2(b)}{b^4 (b\cos(b) - \sin(b))} (\sin(bx) - bx), \tag{38}$$

where $b = \sqrt{q_0}$. Substituting eigenfunctions (33), (34) and associated function (38) into expression (36), we find the normal vector to the flutter boundary at the point $\mathbf{p}_0 = (0.35431330, 17.0695748)$

$$\mathbf{f}_2 = (-24288.8139, -1024.49949).$$

Knowledge of the normal vector allows us to study the neighborhood of the point on the flutter boundary in any direction $\mathbf{e}$ such that $\langle \mathbf{f}_2, \mathbf{e} \rangle \neq 0$. In particular, for two orthogonal directions $\mathbf{e} = (1, 0)$ and $\mathbf{e} = (0, 1)$, we get

$$\lambda = \pm 155.848689 \sqrt{\eta_0 - \eta}, \quad \lambda = \pm 32.0078037 \sqrt{q_0 - q}, \tag{39}$$

appropriately. It is easy to see that in typical situation the double zero eigenvalue splits either into complex-conjugate pair or into two real eigenvalues, one of which is negative, Tab. 1. Thus, the normal vector $\mathbf{f}_2$ at the point $\mathbf{p}_0$ is directed into the divergence domain. The inequality $\langle \mathbf{f}_2, \mathbf{e} \rangle > 0$ defines the tangent cone to this domain, and $\langle \mathbf{f}_2, \mathbf{e} \rangle < 0$ defines the tangent cone to the flutter domain, Fig. 2. Only curves, emitted along the tangent vector to the boundary can lead to stability domain from the singular point. The direction of the appropriate tangent vector $\mathbf{e}_*$ can be found by consideration of bifurcation of double zero eigenvalue in the degenerate case $\langle \mathbf{f}_2, \mathbf{e}_* \rangle = 0$. The answer obtained with the use of the same perturbation technique is $\mathbf{e}_* = (1, -23.7079804)$. It can be seen from the Tab. 1 that the eigenvalue $\lambda_0 = 0$ splits into two positive eigenvalues (stability) only if parameters change in the direction $\mathbf{e}_*$

$$\lambda_1 = 1207.53146\epsilon, \quad \lambda_2 = 43.1083501\epsilon. \tag{40}$$

Note that our technique gives a good approximation to eigenvalues, Tab. 1.

Table 1. SPLITTING OF THE DOUBLE ZERO EIGENVALUE NEAR THE SINGULAR POINT $\mathbf{p}_0 = (0.35431330, 17.0695748)$.

| $(\Delta\eta, \Delta q)$ | $\lambda : Eq.(39), Eq.(40)$ | $\lambda : Eq.(28)$ |
|---|---|---|
| $(0, 10^{-4})$ | $Re\lambda_{1,2}=0$ | $Re\lambda_{1,2}=-0.00151188$ |
| | $Im\lambda_{1,2}=\pm 0.32007804$ | $Im\lambda_{1,2}=\pm 0.32007586$ |
| $(0, -10^{-4})$ | $\lambda_1=0.32007804$ | $\lambda_1=0.32159210$ |
| | $\lambda_2=-0.32007804$ | $\lambda_2=-0.31856833$ |
| $(10^{-4}, 0)$ | $Re\lambda_{1,2}=0$ | $Re\lambda_{1,2}=0.02668744$ |
| | $Im\lambda_{1,2}=\pm 1.55848689$ | $Im\lambda_{1,2}=\pm 1.55823291$ |
| $(-10^{-4}, 0)$ | $\lambda_1=1.55848689$ | $\lambda_1=1.53205170$ |
| | $\lambda_2=-1.55848689$ | $\lambda_2=-1.58543004$ |
| $-10^{-5}\mathbf{e}_*$ | $\lambda_1=-0.01207531$ | $\lambda_1=-0.01207543$ |
| | $\lambda_2=-0.000431084$ | $\lambda_2=-0.000431085$ |
| $10^{-5}\mathbf{e}_*$ | $\lambda_1=0.01207531$ | $\lambda_1=0.01207520$ |
| | $\lambda_2=0.000431084$ | $\lambda_2=0.000431082$ |

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