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MATHEMATICAL PHYSICS

Gyroscopic Stabilization in the Presence of Nonconservative Forces

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This paper analyzes the stability of a linear autonomous nonconservative system with an even number of degrees of freedom in the presence of potential, gyroscopic, dissipative, and nonconservative positional forces. It is well known that, when applied separately, dissipative and nonconservative positional forces destroy gyroscopic stabilization [1, 3]. However, their combination can make a system asymptotically stable. It is found that the complexity of the choice of such a combination is associated with a Whitney umbrella singularity existing on the boundary of the gyroscopic stabilization domain of the nonconservative system. In this paper, an approximation to the boundary of the asymptotic stability domain near the singularity is explicitly found and an analytical estimate of the critical gyroscopic parameter is obtained. As an example, we analyze the stability of Hauger's gyropendulum under the action of a follower torque.

1. Consider an autonomous nonconservative system of the form

$$\ddot{\mathbf{x}} + (\gamma\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + (\nu\mathbf{N} + \mathbf{P})\mathbf{x} = 0, \quad (1)$$

where the dot denotes the time derivative; $\mathbf{x} \in \mathbb{R}^m$; and $\mathbf{D} = \mathbf{D}^T$, $\mathbf{G} = -\mathbf{G}^T$, and $\mathbf{N} = -\mathbf{N}^T$ are real matrices corresponding to dissipative, gyroscopic, and nonconservative positional (circulatory) forces whose magnitudes are controlled by the parameters δ, γ, and ν, respectively. The real matrix $\mathbf{P} = \mathbf{P}^T$ corresponds to potential forces. Seeking a solution to Eq. (1) in the form $\mathbf{x} = \mathbf{u}\exp(\lambda t)$, we arrive at the eigenvalue problem

$$(\mathbf{I}\lambda^2 + (\gamma\mathbf{G} + \delta\mathbf{D})\lambda + \nu\mathbf{N} + \mathbf{P})\mathbf{u} = 0, \quad (2)$$

where $\mathbf{I}$ is the identity matrix, λ is an eigenvalue, and $\mathbf{u}$ is an eigenvector.

In the presence of only potential forces with a matrix $\mathbf{P} = -\mathbf{K} < 0$, the trivial solution to Eq. (1) is statically unstable. In the absence of nonconservative forces ($\delta = \nu = 0$), the phenomenon of gyroscopic stabilization lies in that, for even m, the statically unstable potential system becomes Lyapunov stable due to the action of gyroscopic forces provided that $\det\mathbf{G} \neq 0$ and the absolute value of the gyroscopic parameter is sufficiently large ($|\gamma| > \gamma_0 > 0$) [1, 3]. In the generic case, all the eigenvalues of the matrix polynomial $\mathbf{I}\lambda^2 + \lambda\gamma\mathbf{G} - \mathbf{K}$ are purely imaginary and simple. In the critical case, for example, at $\gamma = \gamma_0$, there exists a pair of double purely imaginary eigenvalues $\lambda = \pm i\omega_0$ with a Jordan chain of length 2, while the remaining eigenvalues are simple and lie on the imaginary axis [10]. For $m = 2$, the exact critical value of the gyroscopic parameter make is

$$\gamma_0 = \sqrt{\mathrm{Tr}\mathbf{K} + 2\sqrt{\det\mathbf{K}}} = \sqrt{\lambda_1(\mathbf{K})} + \sqrt{\lambda_2(\mathbf{K})} \quad (3)$$

(see [14]). Here, $\lambda_1(\mathbf{K})$ and $\lambda_2(\mathbf{K})$ are eigenvalues of the matrix $\mathbf{K} > 0$ and we assumed that $\det\mathbf{G} = 1$. The determination of γ_0 for $m > 2$ is a much more complicated problem requiring modern geometric methods [12].

The Jordan chain for the eigenvalue $i\omega_0$ consists of an eigenvector $\mathbf{u}_0$ and an associated vector $\mathbf{u}_1$ that satisfy the equations

$$(-\mathbf{I}\omega_0^2 + i\omega_0\gamma_0\mathbf{G} - \mathbf{K})\mathbf{u}_0 = 0, \quad (4)$$

$$(-\mathbf{I}\omega_0^2 + i\omega_0\gamma_0\mathbf{G} - \mathbf{K})\mathbf{u}_1 = -(2i\omega_0\mathbf{I} + \gamma_0\mathbf{G})\mathbf{u}_0 \quad (5)$$

(see [10]). The eigenvector $\mathbf{u}_0$ satisfies the orthogonality condition

$$\mathbf{u}_0^*(2i\omega_0\mathbf{I} + \gamma_0\mathbf{G})\mathbf{u}_0 = 0, \quad (6)$$

where the star denotes the Hermite conjugate. Equation (4) and orthogonality condition (6) imply the Rayleigh quotient for the critical frequency:

$$\omega_0^2 = \frac{\mathbf{u}_0^*\mathbf{K}\mathbf{u}_0}{\mathbf{u}_0^*\mathbf{u}_0}. \quad (7)$$

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In the neighborhood of $\gamma = \gamma_0$, the double eigenvalue and the corresponding eigenvector vary according to the formulas [10]

$$i\omega(\gamma) = i\omega_0 \pm i\mu\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0), \quad (8)$$

$$\mathbf{u}(\gamma) = \mathbf{u}_0 \pm i\mu\mathbf{u}_1\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0), \quad (9)$$

where μ^2 is a real number given by

$$\mu^2 = -\frac{2\omega_0^2\mathbf{u}_0^*\mathbf{u}_0}{\gamma_0(\omega_0^2\mathbf{u}_1^*\mathbf{u}_1 + \mathbf{u}_1^*\mathbf{K}\mathbf{u}_1 - i\omega_0\gamma_0\mathbf{u}_1^*\mathbf{G}\mathbf{u}_1 - \mathbf{u}_0^*\mathbf{u}_0)}. \quad (10)$$

For $m = 2$ and $\det\mathbf{G} = 1$, the double eigenvalue

$$i\omega_0 = i\sqrt[4]{\det\mathbf{K}} \quad (11)$$

for $\gamma = \gamma_0$ defined by Eq. (2) has a Jordan chain consisting of an eigenvector $\mathbf{u}_0$ and an associated vector $\mathbf{u}_1$:

$$\mathbf{u}_0 = C\begin{pmatrix} -i\omega_0\gamma_0 + k_{12} \\ -\omega_0^2 - k_{11} \end{pmatrix}, \quad (12)$$

$$\mathbf{u}_1 = -\frac{C}{\omega_0^2 + k_{22}}\begin{pmatrix} 0 \\ i\omega_0(k_{11} - k_{22}) + \gamma_0 k_{12} \end{pmatrix}, \quad (13)$$

where C is an arbitrary constant. The vector $\mathbf{u}_1$ is defined up to a term proportional to $\mathbf{u}_0$. Substituting (12) and (13) into Eq. (10) and taking into account of orthogonality condition (6) gives the expression

$$\mu^2 = \frac{\gamma_0}{2}\frac{(k_{11} + \omega_0^2)(k_{22} + \omega_0^2)}{\gamma_0^2\omega_0^2 + k_{12}^2} = \frac{\gamma_0}{2} > 0. \quad (14)$$

The coefficient $\mu = \sqrt{\frac{\gamma_0}{2}}$ is real-valued and, according to formula (8), for $\gamma > \gamma_0$, the double eigenvalue splits into two simple purely imaginary ones (gyroscopic stabilization). It follows from (8) and (10) that the gyroscopic stabilization of a statically unstable potential system for arbitrary even m follows the same scenario [10].

2. Let us study the influence of dissipative and nonconservative positional forces on gyroscopic stabilization. First, we note that the most interesting in practice is the situation when these forces in system (1) are small as compared with the gyroscopic ones: $\delta \sim \nu \ll \gamma \sim \gamma_0$. The critical gyroscopic parameter $\gamma_{cr}(\delta, \nu)$ on the boundary of the gyroscopic stabilization domain of the nonconservative system is a function of the parameters corresponding to the dissipative and circulatory forces and may be considerably different from γ_0. Moreover, stability is extremely sensitive to the choice of a perturbation, while the balance of forces leading to asymptotic stability is not obvious. This is probably why the effect of small dissipative and nonconservative positional forces on the stability of gyroscopic systems is considered paradoxical [8, 15], although the destabilization paradox is also connected with physical paradoxes, such as the inversion of the tippe top's equilibrium and the tendency of Jellet's egg to spin about its longest axis [11, 13, 15]. Although the study of the influence of dissipative and circulatory forces on the stability of rotors that are statically stable in the absence of spinning has a long history [8], much attention to the gyroscopic stabilization of nonconservative systems has apparently been given only recently [6, 8, 11, 13–15].

To estimate $\gamma_{cr}(\delta, \nu)$, we note that, when the potential system gyroscopically stabilized for $\gamma > \gamma_0$ is perturbed by small dissipative and circulatory forces, the simple eigenvalues $i\omega(\gamma)$ take an increment:

$$\lambda = i\omega + \frac{\omega^2\mathbf{u}^*\mathbf{D}\mathbf{u}\delta - i\omega\mathbf{u}^*\mathbf{N}\mathbf{u}\nu}{\mathbf{u}^*\mathbf{K}\mathbf{u} - \omega^2\mathbf{u}^*\mathbf{u}} + o(\delta, \nu), \quad (15)$$

where $\mathbf{u}(\gamma)$ is the eigenvector corresponding to $i\omega(\gamma)$. Since $\mathbf{D}$ and $\mathbf{K}$ are real symmetric matrices and $\mathbf{N}$ is skew-symmetric, this increment is real-valued. Therefore, in the first approximation in δ and ν, the simple eigenvalue $i\omega(\gamma)$ remains on the imaginary axis if

$$\nu = \beta(\gamma)\delta, \quad (16)$$

where

$$\beta(\gamma) = -i\omega(\gamma)\frac{\mathbf{u}^*(\gamma)\mathbf{D}\mathbf{u}(\gamma)}{\mathbf{u}^*(\gamma)\mathbf{N}\mathbf{u}(\gamma)}. \quad (17)$$

For fixed $\gamma > \gamma_0$, expressions (16) and (17) define (at the origin in the plane of δ and ν) a tangent line to the boundary of the domain where the increment of $i\omega(\gamma)$ is negative. In the generic case, a linear approximation to the boundary of the intersection of all such domains corresponding to the various eigenvalues of the unperturbed gyroscopic system is specified by only two straight lines of form (16).

It follows from expansions (8) that, in the neighborhood of $\gamma = \gamma_0$, all the simple eigenvalues $i\omega(\gamma)$ that remain simple at $\gamma = \gamma_0$ and also the corresponding eigenvectors $\mathbf{u}(\gamma)$ vary slowly with γ in comparison with the pair of eigenvalues that coincide with each other at $\gamma = \gamma_0$. Clearly, the behavior of these two eigenvalues is also of key importance for the stability of the gyroscopic system under small nonconservative perturbations. For this reason, the critical gyroscopic parameter on the boundary of the stability domain of system (1) is estimated by plugging expansions (8) and (9) into formula (17):

$$\beta(\gamma) = -(\omega_0 \pm \mu\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0)) \times \frac{d_1 \mp \mu d_2\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0)}{n_1 \pm \mu n_2\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0)}. \quad (18)$$

The coefficients d_1, d_2, n_1, and n_2 are real and have the form

$$d_1 = \mathrm{Re}(\mathbf{u}_0^*\mathbf{D}\mathbf{u}_0), \quad d_2 = \mathrm{Im}(\mathbf{u}_0^*\mathbf{D}\mathbf{u}_1 - \mathbf{u}_1^*\mathbf{D}\mathbf{u}_0), \tag{19}$$

$$n_1 = \mathrm{Im}(\mathbf{u}_0^*\mathbf{N}\mathbf{u}_0), \quad n_2 = \mathrm{Re}(\mathbf{u}_0^*\mathbf{N}\mathbf{u}_1 - \mathbf{u}_1^*\mathbf{N}\mathbf{u}_0). \tag{20}$$

Formula (18) implies that, for $|\beta - \beta_0| \ll 1$, where

$$\beta_0 = \beta(\gamma_0) = -i\omega_0\frac{\mathbf{u}_0^*\mathbf{D}\mathbf{u}_0}{\mathbf{u}_0^*\mathbf{N}\mathbf{u}_0}, \tag{21}$$

the new critical value of the gyroscopic parameter is given by

$$\gamma_{cr}(\beta) = \gamma_0 + \frac{n_1^2(\beta - \beta_0)^2}{\mu^2(\omega_0 d_2 - \beta_0 n_2 - d_1)^2}, \tag{22}$$

which, after substituting $\beta = \frac{\nu}{\delta}$, becomes

$$\gamma_{cr}(\delta, \nu) = \gamma_0 + \frac{n_1^2(\nu - \beta_0\delta)^2}{\mu^2(\omega_0 d_2 - \beta_0 n_2 - d_1)^2\delta^2} \geq \gamma_0. \tag{23}$$

In the space of δ, ν, and γ, the graph of $\gamma_{cr}(\delta, \nu)$ under the condition $(\mu/n_1)(\omega_0 d_2 - \beta_0 n_2 - d_1)\delta > 0$ encloses a domain where two eigenvalues of the unperturbed gyroscopic system that coincide at $\gamma = \gamma_0$ take a negative real increment due to a nonconservative perturbation. At $m = 2$ and for $m > 2$ under the assumption that, when parameters are chosen from this domain, the remaining purely imaginary eigenvalues of the unperturbed system are shifted to the left half of the complex plane, expression (23) is an approximation to the boundary of the asymptotic stability domain of nonconservative gyroscopic system (1) in the neighborhood of $(0, 0, \gamma_0)$ in the parameter space. Moreover, formula (23) gives a simple estimate for $\gamma_{cr}(\delta, \nu)$.

Note that a similar method was used to compute the critical value in [2] in the case of a distributed system (namely, Beck's column with damping) with the only difference being that $\omega(\gamma)$ and $\mathbf{u}(\gamma)$ were determined numerically. The same method but with the help of exact dependences of eigenvalues and eigenvectors of the unperturbed problem on the gyroscopic parameter was used to analyze the stability of a gyropendulum in [8] and the stability of system (1) with $m = 2$ degrees of freedom in [14]. In contrast to [2, 8, 14], we used approximate analytical expressions for the eigenfrequencies and eigenfunctions (see (8) and (9)) obtained by perturbation theory methods, which allowed us to derive a simple analytical estimate of $\gamma_{cr}(\delta, \nu)$ in the general case for $m \geq 2$.

Remarkably, Eq. (23) has the form $Z = \frac{X^2}{Y^2}$, which is canonical for a surface with a singularity known as Whitney's umbrella, which is typical of the boundary of the asymptotic stability domain of a three-parameter family of real matrices [5, 7, 9]. That is why $\gamma_{cr}(\delta, \nu)$ in (23) is not differentiable at zero and depends only on the ratio $\beta = \frac{\nu}{\delta}$. Thus, the limit $\gamma_{cr}(\delta, \nu)$ at zero is indeterminate and depends on the direction specified by β. For all directions $\beta \neq \beta_0$, the limiting value satisfies $\gamma_{cr}(\beta) > \gamma_0$. This means that the critical gyroscopic parameter varies in jumps under infinitesimal variations in δ and ν, which is known as the destabilization paradox and was repeatedly observed in gyroscopic systems with small dissipative and circulatory forces [4, 7, 8, 11] and in circulatory systems with small dissipative and gyroscopic forces [2, 10, 14, 15]. Apparently, the relation of these jumps in the critical value to the Whitney umbrella singularity on the boundary of the asymptotic stability domain was first established in [7] for the problem of stability of a rotating shaft on an elastic foundation.

Note that formulas (21) and (23) extend the result of [14], which was obtained for nonconservative gyroscopic system (1) at $m = 2$ and $\det\mathbf{G} = 1$, to arbitrary even m and $\det\mathbf{G} \neq 0$. Indeed, for $m = 2$ and $\det\mathbf{G} = 1$, the computation of β_0 by formula (21) with the use of eigenvector (12) gives

$$\beta_0 = \frac{\mathrm{Tr}[(\gamma_0^2 - \omega_0^2)\mathbf{D} - \mathbf{K}\mathbf{D}]}{2\gamma_0}, \tag{24}$$

which agrees with [14]. Substituting the eigen- and associated vectors (12) and (13) into (19) and (20) yields

$$n_1 = -2\gamma_0\omega_0(\omega_0^2 + k_{11}), \tag{25}$$

$$n_2 = -2\gamma_0\frac{\omega_0^2(k_{22} - k_{11}) + k_{12}^2}{\omega_0^2 + k_{22}}, \tag{26}$$

$$d_1 = 2\gamma_0\beta_0(\omega_0^2 + k_{11}), \tag{27}$$

$$\frac{d_2}{2\omega_0} = \frac{(d_{12}k_{12} - d_{22}(\omega_0^2 + k_{11}))(k_{22} - k_{11}) - d_{12}k_{12}\gamma_0^2}{\omega_0^2 + k_{22}}. \tag{28}$$

Taking into account $\gamma_0^2 = \mathrm{Tr}\mathbf{K} + 2\omega_0^2$ and using relations (25)–(28), we calculate the expression in the denominator of (23):

$$\omega_0 d_2 - \beta_0 n_2 - d_1 = -2\omega_0^2(\omega_0^2 + k_{11})\mathrm{Tr}\mathbf{D}. \tag{29}$$

Plugging (14), (25), and (29) into Eq. (23), we find, for $\mathrm{Tr}\mathbf{D} \neq 0$, that

$$\gamma_{\mathrm{cr}}(\nu, \delta) = \gamma_0 + \gamma_0 \frac{2}{(\omega_0 \mathrm{Tr}\mathbf{D})^2} \frac{(\nu - \beta_0\delta)^2}{\delta^2}, \quad (30)$$

where β_0 is given by (24). The condition $\frac{\mu}{n_1}(\omega_0 d_2 - \beta_0 n_2 - d_1)\delta = \frac{\delta\omega_0 \mathrm{Tr}\mathbf{D}}{\sqrt{2\gamma_0}} > 0$ singles out a stable pocket of Whitney's umbrella approximated by Eq. (30). Equations (24) and (30) were also derived from the Routh–Hurwitz conditions in [14].

3. As an example, we consider Hauger's gyropendulum [4], which is an axisymmetric rigid body of mass m hinged at the point O on the axis of symmetry as shown in Fig. 1. The body's moment of inertia about the axis through the point O perpendicular to the axis of symmetry is denoted by I, the body's moment of inertia about the axis of symmetry is denoted by I_0, and the distance between the fastening point and the center of mass is s. The orientation of the pendulum, which is associated with the trihedron $Ox_f y_f z_f$, with respect to the fixed trihedron $Ox_i y_i z_i$ is specified by the angles ψ, θ, and φ. The pendulum experiences the force of gravity $G = mg$ and a follower torque T that lies in the plane of the z_i and z_f coordinate axes. The moment vector makes an angle of $\eta\alpha$ with the axis z_i, where η is a parameter ($\eta \neq 1$) and α is the angle between the z_i and z_f axes. Additionally, the pendulum experiences the restoring elastic moment $R = -r\alpha$ in the hinge and the dissipative moments $B = -b\omega_s$ and $K = -k\dot{\varphi}$, where ω_s is the angular velocity of an auxiliary coordinate system $Ox_s y_s z_s$ with respect to the inertial system and r, b, and k are the corresponding coefficients.

The nonlinear equations of motion derived in [4] have the solution

$$\psi = \theta = 0, \quad \dot{\varphi} = -\frac{T}{k} = -\omega_0. \quad (31)$$

Linearization about this solution with the new variables

$$x_1 = \psi, \quad x_2 = \theta, \quad x_3 = \frac{\dot{\varphi}}{\omega_0} + 1 \quad (32)$$

and the subsequent nondimensionalization yield the system of equations [4]

$$\ddot{x}_1 + \delta\dot{x}_1 - \gamma\dot{x}_2 + \rho x_1 - \nu x_2 = 0, \quad (33)$$

$$\ddot{x}_2 + \gamma\dot{x}_1 + \delta\dot{x}_2 + \nu x_1 + \rho x_2 = 0, \quad (34)$$

$$\dot{x}_3 + \frac{\kappa}{\gamma} x_3 = 0, \quad (35)$$

where the dot denotes the derivative with respect to the new time $\tau = \omega_0 t$ and the dimensionless parameters are given by

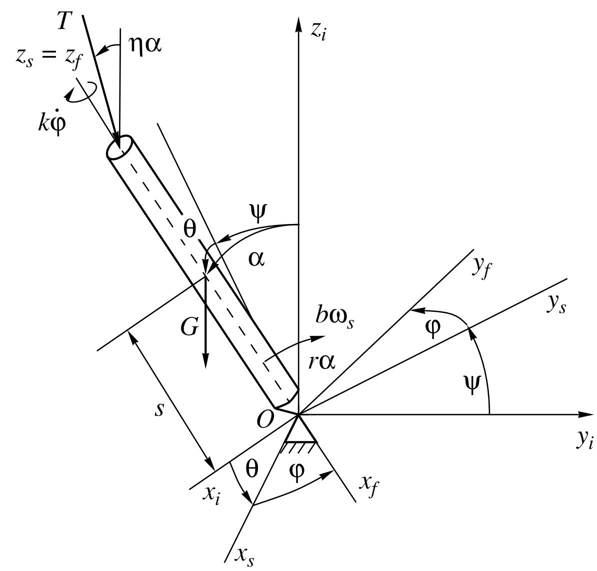

Fig. 1. Hauger's gyropendulum.

$$\gamma = \frac{I_0}{I}, \quad \delta = \frac{b}{I\omega_0}, \quad \kappa = \frac{k}{I\omega_0}, \quad \rho = \frac{r - mgs}{I\omega_0^2}, \quad \nu = \frac{1-\eta}{I\omega_0^2}T. \quad (36)$$

The stability of the pendulum's vertical equilibrium is governed by Eqs. (33) and (34), which are a special case of Eq. (1) at $m = 2$, $\mathbf{D} = \mathbf{I}$, and $\mathbf{P} = \rho\mathbf{I}$. Therefore, they can be written as a single differential equation with complex coefficients:

$$\ddot{x} + i\gamma\dot{x} + \delta\dot{x} + i\nu x + \rho x = 0, \quad (37)$$

where $x = x_1 - ix_2$. This equation, which is also known as the modified Maxwell–Bloch equations, arises in the stability analysis of the tippe top, Jellet's egg, and the Crandall gyropendulum [6, 8, 11, 13, 15].

Applying the Bilharz criterion [10] to the characteristic polynomial $\lambda^2 + (\delta + i\gamma)\lambda + \rho + i\nu$, we find that the trivial solution to Eq. (37) is asymptotically stable if and only if

$$\delta > 0, \quad \gamma > \gamma_{\mathrm{cr}}(\delta, \nu) = \frac{\nu}{\delta} - \frac{\delta}{\nu}\rho. \quad (38)$$

Depending on the sign of ρ, the asymptotic stability domain (38) in the space of three parameters δ, ν, and γ has one of the two typical configurations shown in Fig. 2. For $\rho > 0$, when the restoring elastic moment exceeds the moment produced by gravity, the asymptotic stability domain lies inside a dihedral angle whose

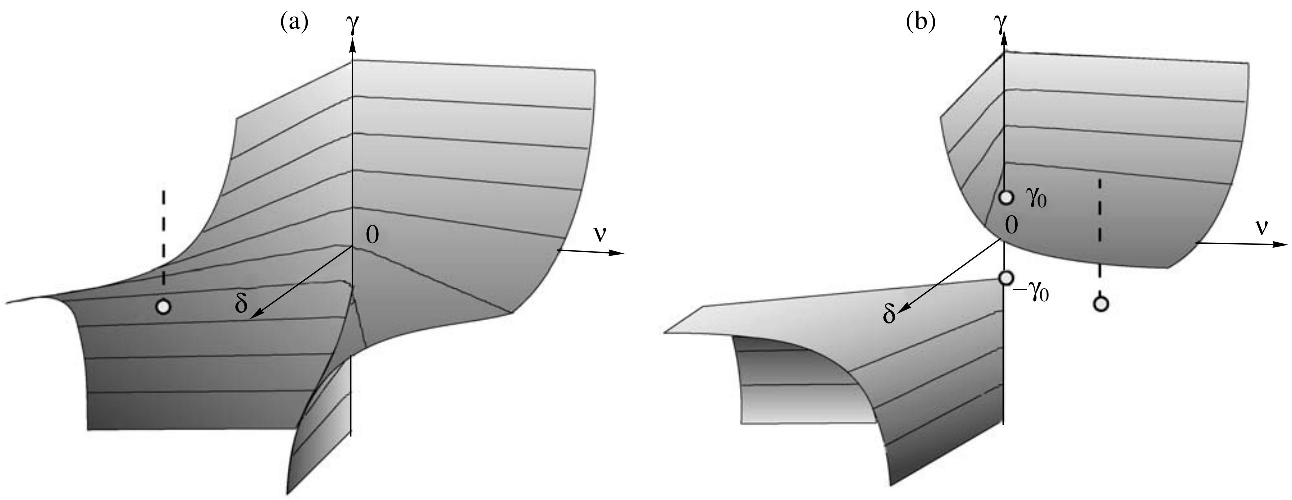

Fig. 2. Asymptotic stability domains of Hauger's gyropendulum: (a) statically stable gyropendulum ($\rho > 0$) and (b) statically unstable gyropendulum ($\rho < 0$).

edge is the γ axis (Fig. 2a). The section of this domain by the plane $\gamma = \text{const}$ is the sector bounded by the lines

$$\nu = \frac{\gamma \pm \sqrt{\gamma^2 + 4\rho}}{2}\delta. \qquad (39)$$

For any γ, the opening angle of the sector is less than π and the asymptotic stability domain twisted about the γ axis lies in the half-space $\delta > 0$. Moreover, the threshold value of γ above which the gyropendulum is unstable is smaller for larger values of ν. Therefore, in the presence of circulatory forces, a system that is stable at $\gamma = 0$ can relatively easily lose its stability with increasing γ as shown in Fig. 2a (dashed line).

As $\rho > 0$ decreases, the hypersurfaces forming the dihedral angle approach each other so that, at $\rho = 0$, they temporarily merge along the line $\nu = 0$, and a new configuration (Fig. 2b) originates for $\rho < 0$. The new domain of asymptotic stability consists of two disjoint parts that are pockets of two Whitney umbrellas singled out by the inequality $\delta > 0$. One of them lies in the domain of positive γ; and the other, in the domain $\gamma < 0$. According to formulas (3) and (11), the singular points on the γ axis correspond to the critical values $\pm\gamma_0 = \pm 2\sqrt{-\rho}$ and the critical frequency $\omega_0 = \sqrt{-\rho}$. Noting that $\gamma_{cr}(\nu = \pm\sqrt{-\rho}\,\delta, \delta) = \pm\gamma_0$ and substituting $\beta = \frac{\nu}{\delta}$ into formula (38), we expand $\gamma_{cr}(\beta)$ in a series in the neighborhood of $\beta = \pm\sqrt{-\rho}$:

$$\gamma_{cr}(\beta) = \pm 2\sqrt{-\rho} \pm \frac{1}{\sqrt{-\rho}}(\beta \mp \sqrt{-\rho})^2 + o((\beta \mp \sqrt{-\rho})^2). \qquad (40)$$

Proceeding from β to ν and δ in (40) yields approximations of the stability boundary near the singularities:

$$\gamma_{cr}(\nu, \delta) = \pm 2\sqrt{-\rho} \pm \frac{1}{\sqrt{-\rho}} \frac{(\nu \mp \delta\sqrt{-\rho})^2}{\delta^2}. \qquad (41)$$

They also follow from formula (30) after substituting $\gamma_0 = 2\sqrt{-\rho}$, $\omega_0 = \sqrt{-\rho}$, and $\beta_0 = \sqrt{-\rho}$, where the last value is given by (24).

Thus, Hauger's gyropendulum, which is unstable at $\gamma = 0$, can become asymptotically stable for sufficiently large $|\gamma| \ge \gamma_0$ under a suitable combination of dissipative and nonconservative positional forces as shown in Fig. 2b (dashed line). Note that, although stability conditions (38) were established in [4], Hauger failed to find Whitney umbrella singularities on the boundary of the pendulum's gyroscopic stabilization domain.

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