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Modeling and Stability Analysis of an Axially Moving Beam With Frictional Contact[1]

This paper considers a moving beam in frictional contact with pads, making the system susceptible for self-excited vibrations. The equations of motion are derived and a stability analysis is performed using perturbation techniques yielding analytical approximations to the stability boundaries. Special attention is given to the interaction of the beam and the rod equations. The mechanism yielding self-excited vibrations does not only occur in moving beams, but also in other moving continua such as rotating plates, for example.
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1 Introduction

In many engineering applications, self-excited vibrations are an unwanted phenomenon. They occur when instabilities arise in a system. There can be various reasons for these instabilities. In this paper, we are particularly interested in self-excited vibrations caused by friction. An example for oscillations of this type is the squealing of disk brakes considered by some of the authors in a previous paper [1], where a discretization approach was used. In the present paper, we give a continuous approach for the traveling beam with clamped boundary conditions that is in frictional contact with two idealized brake pads. An engineering application for the model might be traveling belts, band saws, etc.

Many contributions on axially moving media can be found in the literature. A fundamental work is by Wickert and Mote [2] who investigate the moving string and the moving beam showing mathematical properties and calculating numerically the spectrum of the beam for simply supported and clamped boundary conditions. In a second paper [3], they develop a complex modal analysis for continuous systems using a first-order partial differential equation with respect to time. A similar approach has been developed for continuous systems by Meirovitch in [4], which can also be extended to continuous gyroscopic systems as done in [5]. In [6], Parker investigates the eigenvalues of gyroscopic continua in the vicinity of the critical speeds using perturbation techniques on the first-order system. In particular, he analytically calculates the critical speeds for the simply supported moving beam. In a paper by Seyranian and Kliem [7], the splitting of the double zero eigenvalues at the critical speeds of the beam is investigated using perturbation techniques directly on the operator polynomial.

In all the papers cited above, the boundary conditions were self-adjoint. Only a few authors consider the influence of nonconservative forces on axially moving continua. For example, Cheng and Perkins study the stability of a string sliding through an elastically supported dry friction guide [8]. However, in their model, the friction forces only affect the tension of the string and, therefore, no instability occurs before the first critical speed. A related problem is the stationary beam under moving frictional forces discussed in [9]. In the literature about rotating plates, more papers dealing with nonconservative forces can be found, for example, we mention [10,11].

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For the traveling beam considered in our paper, the frictional contact makes the problem nonconservative and introduces intermediate transition conditions into the boundary value problem. Special attention is given to the coupling of beam and rod equations using the assumptions of the Euler-Bernoulli theory in the linear elasticity problem and taking into account the exact contact kinematics of the beam and the pads.

We obtain a distributed gyroscopic system with dissipative and nonconservative positional forces originating from the pads. Since the gyroscopic stability is highly sensitive to the influence of the dissipative and especially nonconservative positional forces (see [12–22]), the stability analysis needs more sophisticated tools than the ones used in the previous papers. Recently, Kirillov and Seyranian [23–25] developed an effective method of analyzing stability boundaries and its singularities for distributed nonconservative systems based on the bifurcation theory of eigenvalues of two point non-self-adjoint boundary value problems with the differential expression and boundary conditions depending on the spectral parameter and multiple physical parameters. We develop this approach further to study boundary value problems with intermediate transition conditions.

The outline of the paper is as follows. We first derive the model from the theory of linear elasticity using the principle of virtual work. The stability of the system is then investigated by interpreting damping and nonconservative forces as perturbations. We use a discretization approach for the stability analysis of a nonperturbed conservative gyroscopic system; then, based on numerically obtained data, we perform a perturbation analysis directly on the boundary value problem of the nonconservatively loaded beam.

2 Derivation of the Mathematical Model

We consider an axially moving Euler-Bernoulli beam sliding through two idealized massless brake pads with constant velocity $\dot{q}_0$; see Fig. 1. We introduce a spatially fixed frame with unit vectors $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, and $\boldsymbol{e}_z$ and a frame with unit vectors $\boldsymbol{e}_{\tilde{x}}$, $\boldsymbol{e}_{\tilde{y}}$, and $\boldsymbol{e}_{\tilde{z}}$ moving with the undeformed configuration of the beam. The beam is pretensioned with the force κ before applying the pads. As usual in Euler-Bernoulli theory, we neglect the stresses σ_y, σ_z, and τ_{yz}, and assume that the cross sections of the beam stay planar and perpendicular to the neutral plane. The mass of a cross section is assumed to be concentrated on the neutral plane and is assumed to be constant between the two supports. Each point on the neutral plane has a displacement $u(x,t)$ in the x direction and $w(x,t)$ in the z direction counted out of the prestressed configuration with no pads in contact with the beam.

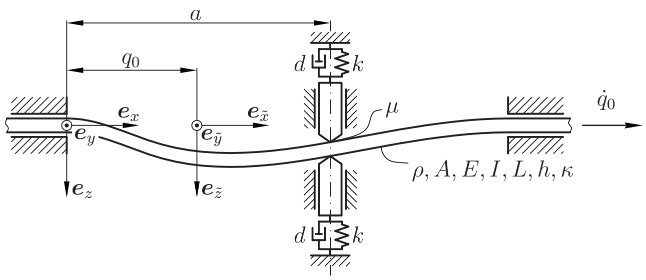


Fig. 1 Axially moving beam

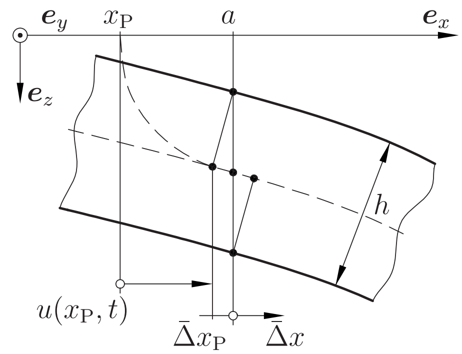


Fig. 2 Contact kinematics

To derive the equations of motion, we use the principle of virtual work with the assumptions stated above giving

$$\int_0^L \int_A \left[\rho \frac{\mathrm{d}^2}{\mathrm{d}t^2} \boldsymbol{p}_\mathrm{M} \cdot \delta \boldsymbol{p}_\mathrm{M} + (\sigma_0 + E\epsilon)\delta\epsilon \right] \mathrm{d}A \, \mathrm{d}x = \sum_i (\boldsymbol{F}_i \cdot \delta \boldsymbol{p}_i) \tag{1}$$

for an extensible Euler-Bernoulli beam, where

$$\epsilon = u' + \frac{1}{2} w'^2 - z w'' \tag{2}$$

is the strain and σ_0 is the pretension of the beam. The forces $\boldsymbol{F}_i$ are the contact forces between the pads and the beam and $\delta \boldsymbol{p}_i$ are the virtual velocities/displacements of the contact points on the beam. In order to calculate the contact forces, we have to consider the contact kinematics.

2.1 Kinematics. A point on the neutral fiber is

$$\boldsymbol{p}_\mathrm{M}(x,t) = [x + u(x,t)]\boldsymbol{e}_x + w(x,t)\boldsymbol{e}_z \tag{3}$$

where $x = q_0(t)$ due to the kinematic constraint. When differentiating $\boldsymbol{p}_\mathrm{M}(x,t)$ with respect to time, we therefore have

$$\frac{\mathrm{d}}{\mathrm{d}t} \boldsymbol{p}_\mathrm{M}(x,t) = [\dot{q}_0 + u_t(x,t) + \dot{q}_0 u_x(x,t)]\boldsymbol{e}_x + [w_t(x,t) + \dot{q}_0 w_x(x,t)]\boldsymbol{e}_z \tag{4}$$

Since cross sections stay planar and perpendicular to the neutral plane, it is possible to describe points on the upper surface of the beam through points on the neutral fiber, which is parametrized by

$$f(x,z,t) = z - w(x,t) = 0 \tag{5}$$

The position vector of a point on the upper surface of the beam is given by

$$\boldsymbol{p}(x,t) = [x + u(x,t)]\boldsymbol{e}_x + w(x,t)\boldsymbol{e}_z - \frac{h}{2}\boldsymbol{e}_\nabla(x,t) \tag{6}$$

where

$$\boldsymbol{e}_\nabla(x,t) = \frac{\nabla f(x,z,t)}{|\nabla f(x,z,t)|} \tag{7}$$

is the gradient vector to the point x on the neutral plane. The position vector of the point currently in contact with the upper pad is given by

$$(x_\mathrm{P} + u(x_\mathrm{P},t))e_x + w(x_\mathrm{P},t)\boldsymbol{e}_z - \frac{h}{2}\boldsymbol{e}_\nabla(x_\mathrm{P},t) \tag{8}$$

where since we work in Lagrangian coordinates $x_\mathrm{P} = a + \Delta x_\mathrm{P}$ is the position of the point on the neutral plane corresponding to the point currently in contact with the upper pad, as shown in Fig. 2.

From geometrical considerations, it is seen that $\Delta x_\mathrm{P} = \bar{\Delta} x_\mathrm{P} + u(a + \Delta x_\mathrm{P}, t)$, where

$$\bar{\Delta} x_\mathrm{P} = -\frac{h}{2}\sin[\arctan w'(a + \bar{\Delta} x_\mathrm{P}, t)] \tag{9}$$

$$= -\frac{h}{2} \frac{w'(a + \bar{\Delta} x_\mathrm{P}, t)}{\sqrt{1 + w'(a + \bar{\Delta} x_\mathrm{P}, t)^2}} \tag{10}$$

that is a fixed point equation of the type

$$\bar{\Delta} x_\mathrm{P}^{k+1} = g(\bar{\Delta} x_\mathrm{P}^k) \quad \bar{\Delta} x_\mathrm{P}^0 = 0 \tag{11}$$

Since $|w'(x,t)| \ll 1$ we obtain

$$|g(x) - g(y)| = \left| \frac{h}{2} \frac{w'(x,t)}{\sqrt{1 + w'(x,t)^2}} - \frac{h}{2} \frac{w'(y,t)}{\sqrt{1 + w'(y,t)^2}} \right| < 1 \tag{12}$$

and therefore the mapping g is contracting. The Banach fixed point theorem is therefore applicable and $\bar{\Delta} x_\mathrm{P}$ can be iteratively determined to arbitrary precision. The quantity Δx_P is now determined from $\Delta x_\mathrm{P} = \bar{\Delta} x_\mathrm{P} + u(a + \Delta x_\mathrm{P}, t)$, which is also a fixed point equation that can be solved using the Banach fixed point theorem. For the lower contact point, we proceed similarly. The position vectors and, hence, the virtual velocities of the contact points, can thus be determined to arbitrary precision.

2.2 Contact Forces. The contact forces between the surface of the beam and the pads have already been stated in [1], but are restated here for the convenience of the reader.

The normal force is perpendicular to the surface of the beam and is therefore given by:

$$\boldsymbol{N}_\mathrm{P} = -\boldsymbol{N}_{\bar{\mathrm{P}}} = N_\mathrm{P} \boldsymbol{e}_\nabla(x_\mathrm{P}, t) \tag{13}$$

We assume the applicability of Coulomb's law of friction, and therefore the friction force has the magnitude $R_\mathrm{P} = \mu N_\mathrm{P}$, and is directed against the relative velocity between the point P on the beam and the point $\bar{\mathrm{P}}$ on the pad (see Fig. 3), so that

$$\boldsymbol{R}_\mathrm{P} = -\boldsymbol{R}_{\bar{\mathrm{P}}} = R_\mathrm{P} \frac{\boldsymbol{v}_{\bar{\mathrm{P}}} - \boldsymbol{v}_\mathrm{P}}{|\boldsymbol{v}_{\bar{\mathrm{P}}} - \boldsymbol{v}_\mathrm{P}|} \tag{14}$$

Throughout the paper we exclude stick slip, which in the linear case, is assured by the condition

$$|\boldsymbol{v}_{\bar{\mathrm{P}}} - \boldsymbol{v}_\mathrm{P}| > 0 \Rightarrow |\dot{u}(a,t)| + |\dot{w}'(a,t)| \frac{h}{2} < \dot{q}_0 \tag{15}$$

From a force balance at the upper pad

$$0 = N_\mathrm{P} \cos \varphi - R_\mathrm{P} \sin \varphi - N_0 + k z_{\bar{\mathrm{P}}} + d \dot{z}_{\bar{\mathrm{P}}} \tag{16}$$

where $z_{\bar{\mathrm{P}}}$ is the vertical displacement of the point $\bar{\mathrm{P}}$ on the pad, with

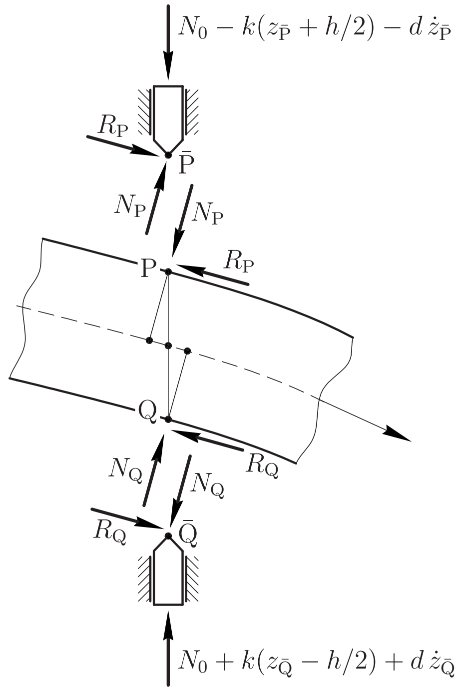


Fig. 3 Contact forces

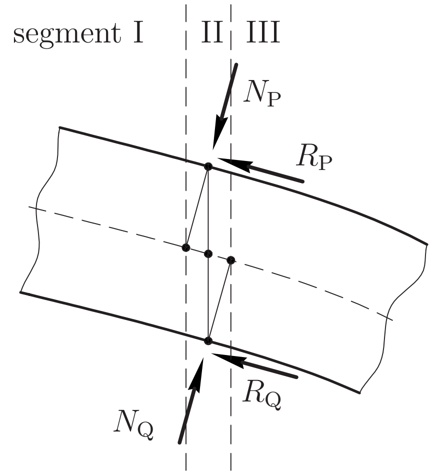


Fig. 4 Segments of the beam

$$\varphi = \arctan[w'(x_P,t)] \tag{17}$$

we can now calculate the magnitude of N_P, so that the contact forces are completely defined. Since in Euler-Bernoulli theory the cross sections of the beam stay planar, we can replace $\boldsymbol{N}_P$ and $\boldsymbol{R}_P$ by an equivalent system of loads consisting of the force $B_P\boldsymbol{e}_x + A_P\boldsymbol{e}_z$ and a torque M_P, both acting on the neutral fiber. Again, we can proceed similarly for the lower contact point Q. The linearized expressions for the forces and torques are then

$$B_P = -\mu(N_0 - kw(x_P,t) - d\dot{w}(x_P,t)) - N_0(1+\mu^2)w'(x_P,t) \tag{18}$$

$$A_P = N_0 - kw(x_P,t) - d\dot{w}(x_P,t) \tag{19}$$

$$M_P = \frac{1}{2}h[\mu^2 N_0 w'(x_P,t) + \mu N_0 - k\mu w(x_P,t) - d\mu\dot{w}(x_P,t)] \tag{20}$$

$$B_Q = -\mu[N_0 + kw(x_Q,t) + d\dot{w}(x_Q,t)] + N_0(1+\mu^2)w'(x_Q,t) \tag{21}$$

$$A_Q = -N_0 - kw(x_Q,t) - d\dot{w}(x_Q,t) \tag{22}$$

$$M_Q = \frac{1}{2}h[\mu^2 N_0 w'(x_Q,t) - \mu N_0 - k\mu w(x_Q,t) - d\mu\dot{w}(x_Q,t)] \tag{23}$$

2.3 Boundary Value Problem. Due to the contact forces, the derivatives w'' and w''' will not be continuous at the points x_P and x_Q, i.e., their left and right limits do not coincide; for example $w''(x_P^-,t) \neq w''(x_P^+,t)$. Therefore, we have to consider three different segments of the beam as shown in Fig. 4.

Carrying out the variations in (1) requiring that the functions satisfy the geometric boundary conditions and applying the main theorem of variational calculus, we obtain a boundary value problem for the beam in w

$$\rho A\left[\ddot{w}(x,t) + 2\dot{q}_0\dot{w}'(x,t) + \left(\dot{q}_0^2 - \frac{\kappa}{\rho A}\right)w''(x,t)\right] + EIw^{IV}(x,t) = 0 \tag{24}$$

with boundary conditions

$$w(0,t) = w(L,t) = 0 \quad w'(0,t) = w'(L,t) = 0 \tag{25}$$

and transition conditions

$$A_{P,Q} + EI[w'''(x_{P,Q}^-,t) - w'''(x_{P,Q}^+,t)] = 0 \tag{26}$$

$$M_{P,Q} + EI[w''(x_{P,Q}^-,t) - w''(x_{P,Q}^+,t)] = 0 \tag{27}$$

Furthermore, we get a boundary value problem for the rod in u, which consists of the partial differential equation

$$\rho A\ddot{u}(x,t) + 2\dot{q}_0\rho A\dot{u}'(x,t) + (\dot{q}_0^2\rho A - EA)u''(x,t) = 0 \tag{28}$$

with boundary conditions

$$u(0,t) = u(L,t) = 0 \tag{29}$$

and transition conditions

$$EA[u'(x_{P,Q}^-,t) - u'(x_{P,Q}^+,t)] - B_{P,Q} = 0 \tag{30}$$

Two important facts are to be noted: first of all, from Eqs. (19) and (20) we observe that u does not occur in the boundary value problem of the beam and the beam equations can be solved independently of the boundary value problem for the rod. However, vibrations of the beam very well excite the rod as is seen from (18) and (21). That means the stability behavior of the system is determined by the beam equations, at least beyond the critical speed for the rod. At a first view, this one-sided coupling might be surprising, since in conservative problems, this phenomenon cannot occur. The present problem is, however, nonconservative due to the friction forces and it can be seen that the coupling vanishes for $\mu=0$. The second fact to be noted is that the boundary value problems are nonconservative because of the transition conditions through which the contact forces between beam and pads enter the system.

Since segment II of the beam is very small, it is possible to simplify the transition conditions by expanding terms containing x_P or x_Q around $x=a$; for example

$$u'(x_P^+,t) = u'(a,t) + \mathcal{O}(w^2) \tag{31}$$

The transition conditions at x_P and x_Q then simplify to a single transition condition at $x=a$. The transition conditions (26)–(30) are thus replaced by

$$A + EI[w'''(a^-,t) - w'''(a^+,t)] = 0 \tag{32}$$

$$M + EI[w''(a^-,t) - w''(a^+,t)] = 0 \tag{33}$$

$$B - EA[u'(a^-,t) - u'(a^+,t)] = 0 \tag{34}$$

where $A = A_P + A_Q = -2kw(a,t) - 2d\dot{w}(a,t)$, $M = M_P + M_Q = h[N_0\mu^2 w'(a,t) - k\mu w(a,t) - d\mu\dot{w}(a,t)]$ and $B = B_P + B_Q = -2\mu N_0$. The simplification of the transition conditions therefore leads to a complete uncoupling of the boundary value problems for the beam and the rod.

2.4 Discretization. In [1], the authors investigated the traveling beam using a Ritz discretization approach. We will use these discretized equations and compare them to the results obtained from the continuous approach taken in this paper. Using the Ritz expansion in (1)

$$w(x,t) = \sum_{i=1}^{I} W_i(x) q_i(t) \tag{35}$$

yields nonlinear equations of motion $\ddot{\boldsymbol{q}} = f(\boldsymbol{q}, \dot{\boldsymbol{q}})$, which can be linearized to

$$\boldsymbol{M}\ddot{\boldsymbol{q}} + (\boldsymbol{G} + \boldsymbol{D})\dot{\boldsymbol{q}} + \boldsymbol{K}\boldsymbol{q} = \boldsymbol{0} \quad \boldsymbol{M} = \boldsymbol{M}^T \quad \boldsymbol{G} = -\boldsymbol{G}^T$$

$$\boldsymbol{D} \neq \boldsymbol{D}^T \quad \boldsymbol{K} \neq \boldsymbol{K}^T \tag{36}$$

where

$$m_{ji} = \rho A \int_0^L W_j W_i \mathrm{d}x \tag{37}$$

$$g_{ji} = \rho A \dot{q}_0 \int_0^L (W_j W_i' - W_j' W_i)\mathrm{d}x$$

$$d_{ji} = d[2W_j(a)W_i(a) + h\mu W_j'(a)W_i(a)] \tag{38}$$

$$k_{ji} = (\kappa - \rho A \dot{q}_0^2)\int_0^L W_j' W_i' \mathrm{d}x + EI\int_0^L W_j'' W_i'' \mathrm{d}x + 2kW_j(a)W_i(a) - hk\mu W_j'(a)W_i(a) + hN_0(1+\mu^2)W_j'(a)W_i'(a) - \frac{h^2 N_0 \mu}{2} W_j''(a)W_i'(a) \tag{39}$$

Using a result of Karapetjan [16] and Lakhadanov [19] in [1], it was concluded that in the undamped case $d=0$, the stability domain of the nonconservative gyroscopic system is a set of measure zero in the space of the system parameters. Provided that the Ritz expansion converges to the solution, which is the case choosing appropriate shape functions, this result carries over to the continuous problem. It would now be feasible to perform a perturbation analysis of these discretized equations. However, we prefer to perform a perturbation analysis directly on the continuous system, using the discretization only to calculate the spectrum of the unperturbed problem.

Before continuing with the investigation of the unperturbed problem, we draw the reader's attention to a difference between the continuous approach used to derive the boundary value problem, and the Ritz discretization approach taken from [1]. From the derivation of the simplified boundary value problem in Sec. 2.3, it is clear that, since $\Delta x_{P,Q}$ occur only in the arguments of u and w, and are expandable in u and w without having a constant part, they do in fact not enter the simplified boundary value problem. However, when considering the virtual work of the contact forces in the Ritz expansion, we calculated

$$\delta W_A = A_P \sum_{i=1}^{I} W_i(a + \Delta x_P)\,\delta q_i(t) + A_Q \sum_{i=1}^{I} W_i(a + \Delta x_Q)\,\delta q_i(t) \tag{40}$$

where we expanded

$$W_i(a + \Delta x_P) = W_i(a) + W_i'(a)\Delta x_P + \cdots = W_i(a) + W_i'(a)\frac{h}{2}\sum_{j=1}^{I}(W_j'(a)q_j + \mathcal{O}(q_j^2)) \tag{41}$$

without considering the stretching of the beam. Due to the constant terms in A_P and A_Q, we get terms of the form $hN_0 W_j'(a)W_i'(a)$ in the discretized equation of motion, that would not have shown up neglecting $\Delta x_{P,Q}$. Similar terms arise from $M_{P,Q}$. The explanation for this lies in the fact that in the Ritz discretization, the energy expressions were considered up to second order in the q_i, whereas to derive the boundary value problem, a purely geometric linearization was performed with respect to w and u. To get comparable results from the perturbation approach on the discretized system, it would therefore be appropriate to neglect the influence of $\Delta x_{P,Q}$ and perform a geometric linearization. Since we only need the discretization for the unperturbed problem, the corresponding equations are not stated separately.

3 Stability Analysis

In this section, we perform a stability analysis of the beam with the simplified transition conditions (32) and (33). Since the transverse vibrations of the beam are of major interest in applications, we concentrate on their investigation.

We assume that as in many squeal problems, the friction and damping forces coming from the pads are small compared to inertia, gyroscopic, and restoring terms. Therefore, we multiply all forces coming from the pads with δ if they come from damping in the pad, and with γ otherwise, with γ and δ serving as weights for their contribution. Introducing the dimensionless time $\tau = \omega t$ and length $\bar{x} = x/L$ where $\omega^2 = EI/\rho A L^4$ yields the dimensionless parameters

$$\bar{a} = \frac{a}{L} \quad \bar{\rho} = \frac{\dot{q}_0}{L\omega} \quad \bar{h} = \frac{h}{L} \quad \bar{k} = \frac{kL^3}{EI} \quad \bar{d} = \frac{\omega d L^3}{EI} \quad \bar{N}_0 = \frac{N_0 L^2}{EI}$$

$$\bar{\kappa} = \frac{\kappa L^2}{EI} \tag{42}$$

Using the ansatz $w(\bar{x},t) = w(\bar{x})e^{\lambda t}$ where, after separation of time, we use the same symbol w for notational simplicity, the boundary value problem can now be stated as

$$L(w) = \lambda^2 w + 2\bar{\rho}\lambda w' + (\bar{\rho}^2 - \bar{\kappa})w'' + w^{\mathrm{IV}} = 0 \tag{43}$$

where L is a linear differential operator with boundary and transition conditions

$$U_1(w) = U_1^0(w) + \gamma U_1^{1\gamma}(w) + \delta U_1^{1\delta}(w) = 0, \ldots, U_8(w) = U_8^0(w) + \gamma U_8^{1\gamma}(w) + \delta U_8^{1\delta}(w) = 0 \tag{44}$$

where

$$U_i^0(w) = \alpha_0 w(0) + \alpha_1 w'(0) + \cdots + \alpha_7 w'''(1)$$

$$U_i^{1\gamma,\delta}(w) = \beta_0^{\gamma,\delta} w(\bar{a}^-) + \beta_1^{\gamma,\delta} w'(\bar{a}^-) + \cdots + \beta_7^{\gamma,\delta} w'''(\bar{a}^+) \tag{45}$$

are linear forms in $w(\bar{x})$ and its derivatives taken at $\bar{x}=0$, $\bar{x}=1$, $\bar{x}=\bar{a}^-$, and $\bar{x}=\bar{a}^+$. In our case, they read

$$U_1^0 = w(0) \quad U_2^0 = w'(0) \quad U_3^0 = w(\bar{a}^-) - w(\bar{a}^+) \tag{46}$$

$$U_4^0 = w'(\bar{a}^-) - w'(\bar{a}^+) \quad U_5^0 = w''(\bar{a}^-) - w''(\bar{a}^+) \quad U_6^0 = w'''(\bar{a}^-) - w'''(\bar{a}^+) \tag{47}$$

$$U_7^0 = w(1) \quad U_8^0 = w'(1) \tag{48}$$

for the unperturbed problem, and the perturbations are given by

$$U_5^{1\gamma} = -\bar{h}\bar{k}\mu w(\bar{a}) + \bar{h}\bar{N}_0\mu^2 w'(\bar{a}) \quad U_5^{1\delta} = -\bar{h}\bar{d}\mu\lambda w(\bar{a}) \tag{49}$$

$$U_6^{1\gamma} = -2\bar{k}w(\bar{a}) \quad U_6^{1\delta} = -2\bar{d}\lambda w(\bar{a}) \tag{50}$$

other forms $U_i^{1\gamma,\delta}$ being zero.

We now derive the boundary value problem adjoint to (43) and (44). We multiply the differential expression $L(w)$ by a function $v(\bar{x})$ and integrate over the intervals $(0,\bar{a})$ and $(\bar{a},1)$ since we are planning to use integration by parts and some of the derivatives of $w(\bar{x})$ are not continuous at $\bar{x}$. Using the notation $\langle w,v\rangle = \int_0^{\bar{a}^-} w\bar{v}d\bar{x} + \int_{\bar{a}^+}^1 w\bar{v}d\bar{x}$, $\bar{v}$ being the complex conjugate to v,we obtain

$$\langle L(w),v\rangle = \langle w,L^*(v)\rangle + \sum_{i=1}^{16} U_i(w)V_{17-i}(v) \tag{51}$$

where we expressed the boundary terms and terms occurring at $\bar{x}=\bar{a}^-$ and $\bar{x}=\bar{a}^+$ in terms of linear forms $U_1,\ldots,U_8$, which are, in fact, the boundary and transition conditions, and supplementary forms $U_9,\ldots,U_{16}$, such that we can express all of the boundary and transition terms uniquely in the U_i.

From (51) we obtain the differential equation for the adjoint problem

$$L^*(v) = \bar{\lambda}^2 v - 2\bar{\rho}\bar{\lambda}v' + (\bar{\rho}^2 - \bar{\kappa})v'' + v^{\mathrm{IV}} = 0 \tag{52}$$

with boundary conditions

$$V_1(v) = V_1^0(v) + \gamma V_1^{1\gamma}(v) + \delta V_1^{1\delta}(v) = 0, \ldots, \; V_8(v) = V_8^0(v) + \gamma V_8^{1\gamma}(v) + \delta V_8^{1\delta}(v) = 0$$

where

$$V_1^0 = v(1) \quad V_2^0 = -v'(1) \quad V_3^0 = v(\bar{a}^-) - v(\bar{a}^+)$$

$$V_4^0 = -v'(\bar{a}^-) + v'(\bar{a}^+) \tag{53}$$

$$V_5^0 = (\bar{\rho}^2 - \bar{\kappa})[v(\bar{a}^-) - v(\bar{a}^+)] + v''(\bar{a}^-) - v''(\bar{a}^+) \tag{54}$$

$$V_6^0 = -(\bar{\rho}^2 - \bar{\kappa})[v'(\bar{a}^-) - v'(\bar{a}^+)] - [v'''(\bar{a}^-) - v'''(\bar{a}^+)] + 2\bar{\rho}\lambda[v(a^-) - v(a^+)] \tag{55}$$

$$V_7^0 = -v(0) \quad V_8^0 = v'(0) \tag{56}$$

and $V_i^{1\gamma,\delta}$ are skipped since for our analysis, only the adjoint to the unperturbed problem is required. The supplementary expressions read

$$V_9^0 = (\bar{\rho}^2 - \bar{\kappa})v(1) + v''(1) \tag{57}$$

$$V_{10}^0 = -(\bar{\rho}^2 - \bar{\kappa})v'(1) - v'''(1) + 2\bar{\rho}\lambda v(1) \quad V_{11}^0 = v(\bar{a}^+) \tag{58}$$

$$V_{12}^0 = -v'(\bar{a}^+) \quad V_{13}^0 = (\bar{\rho}^2 - \bar{\kappa})v(\bar{a}^+) + v''(\bar{a}^+) \tag{59}$$

$$V_{14}^0 = -(\bar{\rho}^2 - \bar{\kappa})v'(\bar{a}^+) - v'''(\bar{a}^+) + 2\bar{\rho}\lambda v(a^+) \tag{60}$$

$$V_{15}^0 = -(\bar{\rho}^2 - \bar{\kappa})v(0) - v''(0)$$

$$V_{16}^0 = (\bar{\rho}^2 - \bar{\kappa})v'(0) + v'''(0) + 2\bar{\rho}\lambda v(0) \tag{61}$$

3.1 Spectrum of the Unperturbed Problem. The unperturbed problem $\gamma=\delta=0$ is similar to the problem studied by Wickert and Mote in [2]. It can be written as an operator polynomial

$$L(u) = \lambda^2 M(u) + \lambda G(u) + K(u) = \lambda^2 u + 2\bar{\rho}\lambda u' + (\bar{\rho}^2 - \bar{\kappa})u'' + u^{\mathrm{IV}} = 0 \tag{62}$$

$$M = 1 \quad G = 2\bar{\rho}\frac{\partial}{\partial\bar{x}} \quad K = (\bar{\rho}^2 - \bar{\kappa})\frac{\partial^2}{\partial\bar{x}^2} + \frac{\partial^4}{\partial\bar{x}^4} \tag{63}$$

with boundary conditions

$$u(0) = u(1) = 0 \quad u'(0) = u'(1) = 0 \tag{64}$$

We now use the function u for the unperturbed problem, which is not to be confused with the axial displacement appearing in the rod equations. Note that we consider clamped boundary conditions in contrast to the simply supported boundary conditions studied in [2,6,7].

Because of the boundary conditions (64) the operators M and K are symmetric and the quantities $\langle M(u),u\rangle$ and

$$\langle K(u),u\rangle = \int_0^1 [(\bar{\rho}^2 - \bar{\kappa})u'' + u^{IV}]\bar{u}\mathrm{d}\bar{x}$$

$$= \int_0^1 [-(\bar{\rho}^2 - \bar{\kappa})u'\bar{u}' + u''\bar{u}'']\mathrm{d}\bar{x} \tag{65}$$

are real, where $\bar{u}$ is the complex conjugate of u. The operator $G(u)$ is skew symmetric i.e., $\langle G(u),u\rangle = -\langle u,G(u)\rangle$ is a purely imaginary quantity.

The eigenvalues of the unperturbed boundary value problem are found setting $u(\bar{x})=e^{\beta\bar{x}}$, which yields four different solutions β_i depending on λ. Hence the eigenfunctions of the unperturbed problem are given by

$$u(\bar{x}) = A_1 e^{\beta_1\bar{x}} + A_2 e^{\beta_2\bar{x}} + A_3 e^{\beta_3\bar{x}} + A_4 e^{\beta_4\bar{x}} \tag{66}$$

where the A_i are constants. To determine the A_i, we substitute $u(\bar{x})$ into the four boundary conditions, which yields a linear homogeneous system of equations $\boldsymbol{U}(\lambda)\boldsymbol{A}=\boldsymbol{0}$. For nontrivial solutions, the determinant of the coefficient matrix has to vanish. The values λ thus obtained are the eigenvalues of the problem (see [26], [27]). Having calculated the eigenvalues, the eigenfunctions $u(\bar{x})$ can be calculated. Eigenfunctions corresponding to different eigenvalues are linearly independent. If an eigenvalue of multiplicity m occurs and there are $p\leq m$ linearly independent eigenfunctions, then to every eigenfunction $u^k(\bar{x})$, $k=1,\ldots,p$ corresponds a Jordan (Keldysh) chain $u^k \to u_1^k,\ldots,u_{m_k}^k$ of linearly independent associated functions $u_1^k(\bar{x}),\ldots,u_{m_k}^k(\bar{x})$ defined by

$$L(u^k) = 0 \tag{67}$$

$$L(u_1^k) + \frac{1}{1!}\frac{\partial L}{\partial\lambda}(u^k) = 0 \tag{68}$$

$$\ldots \tag{69}$$

$$L(u_{m_k}^k) + \frac{1}{1!}\frac{\partial L}{\partial\lambda}(u_{m_k-1}^k) + \cdots + \frac{1}{m_k!}\frac{\partial^{m_k}L}{\partial\lambda^{m_k}}(u^k) = 0 \tag{70}$$

with $m_1+\cdots+m_p=m-p$ [27].

Taking the scalar product of $L(u)$ with an eigenfunction of the adjoint problem, which is easily seen to be $v=u$, for purely imaginary eigenvalues λ we obtain

$$\langle L(u),u\rangle = \lambda^2\langle M(u),u\rangle + \lambda\langle G(u),u\rangle + \langle K(u),u\rangle = 0 \tag{71}$$

from which we get (see e.g., [15])

$$\lambda = \frac{-\langle G(u),u\rangle \pm \sqrt{\Delta}}{2\langle M(u),u\rangle} \tag{72}$$

$$\Delta = \langle G(u),u\rangle^2 - 4\langle K(u),u\rangle\langle M(u),u\rangle \tag{73}$$

Note that only one of the λ in (72) is an eigenvalue of the system. It can, however, still be seen from (72) that on the divergence boundary $\langle K(u),u\rangle=0$ holds, and that on the flutter boundary we

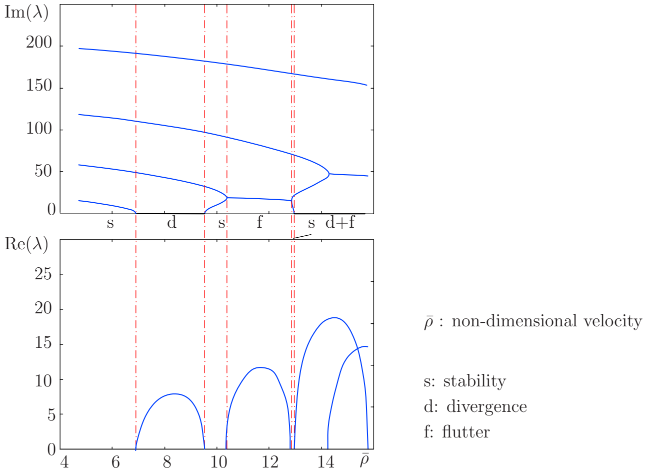


Fig. 5 Eigenvalue curves for $\bar{\kappa}=3\pi$

have $\Delta=0$. Note that for $\Delta=0$, it follows from (72) that

$$2\langle M(u),u\rangle\lambda+\langle G(u),u\rangle=0 \tag{74}$$

which is, in fact, the necessary and sufficient condition for existence of the associated function u_1 following from (68), and its analog is known in aero-elasticity problems [28] as flutter condition. This reflects the fact (see e.g., [29]) also clear from the perturbation formulas derived in Secs. 3.2.1 and 3.2.2 that generically on the stability boundary of a gyroscopic system, we always find a double eigenvalue with a Jordan chain [30].

We can also observe the fact known for discrete systems (see e.g., [15]) that the gyroscopic system has to pass the divergence boundary before it can experience flutter, since from (72) it is seen that flutter can only occur with a negative definite stiffness operator.

Since the characteristic equation is highly nonlinear, we prefer to use the discretization approach of the previous section to calculate the eigenvalues of the problem. With $\delta=\gamma=0$, we obtain the eigenvalue curves shown in Fig. 5, which qualitatively agree with the curves obtained in [2].

In the special case that $\bar{\kappa}=0$, we can calculate the critical speeds analytically by substituting $\bar{\kappa}=0$ in (62) and using (66). For nontrivial solutions, we obtain the condition

$$0=\sin\frac{\bar{\rho}}{2}\left(\sin\frac{\bar{\rho}}{2}-\frac{\bar{\rho}}{2}\cos\frac{\bar{\rho}}{2}\right) \tag{75}$$

which is satisfied for $\bar{\rho}=2\pi n$ and $\bar{\rho}/2=\tan\bar{\rho}/2$. At the first critical speed $\bar{\rho}=2\pi$, we have a double zero eigenvalue with a Jordan chain. For $\bar{\kappa}=0$, we can also analytically obtain the corresponding eigenfunction

$$u(\bar{x})=C_1[1-\cos(2\pi\bar{x})] \tag{76}$$

and the associated function

$$u_1(\bar{x})=C_2\left(1+\frac{C_1}{2\pi}-\cos(\pi\bar{x})-\frac{5}{2\pi^2}\sin(\pi\bar{x})\right) \tag{77}$$

where C_1 and C_2 are undetermined constants. For $\bar{\kappa}\neq 0$, we proceed similarly; the equations are lengthier and are not stated here.

3.2 Perturbation Analysis of the Nonconservative Problem. To analyze the stability of the gyroscopic system with dissipative and nonconservative positional forces, we use the approach of [23–25] based on the perturbation theory of Vishik and Lyusternik [31]. We will study the stability domains for the system (43), (44) depending on the parameters $\bar{\rho}$, δ, and γ. We perturb the system around a fixed speed i.e., $\bar{\rho}=\tilde{\rho}+\nu$ and assume that the small parameters ν, δ, and γ are smooth functions of the parameter ε. This corresponds to a variation along a smooth curve parameterized by ε in the parameter space. It is possible to expand $\nu(\varepsilon)$, $\delta(\varepsilon)$, and $\gamma(\varepsilon)$around $\varepsilon=0$, assuming that $\nu(0)=\delta(0)=\gamma(0)=0$, for example,

$$\gamma(\varepsilon)=\frac{\mathrm{d}\gamma(0)}{\mathrm{d}\varepsilon}\varepsilon+\cdots=\gamma_1\varepsilon+\cdots \tag{78}$$

$$\delta(\varepsilon)=\frac{\mathrm{d}\delta(0)}{\mathrm{d}\varepsilon}\varepsilon+\cdots=\delta_1\varepsilon+\cdots \tag{79}$$

$$\nu(\varepsilon)=\frac{\mathrm{d}\nu(0)}{\mathrm{d}\varepsilon}\varepsilon+\cdots=\nu_1\varepsilon+\cdots \tag{80}$$

Assuming this kind of perturbation, we write the boundary value problem perturbed up to first order in ε as

$$L(w)+\varepsilon L^{1\varepsilon}(w)=0 \tag{81}$$

$$U_1(w)=U_1^0(w)+\varepsilon U_1^\varepsilon(w)=0,\ \ldots,U_8(w)=U_8^0(w)+\varepsilon U_8^\varepsilon(w)=0 \tag{82}$$

where $L(w)$ is defined by (43):

$$L^{1\varepsilon}(w)=\nu_1(2\lambda w'+2\tilde{\rho}w'') \tag{83}$$

$U_i^0(w)$ are defined by (46)–(48) and

$$U_i^{1\varepsilon}=\gamma_1 U_i^{1\gamma}+\delta_1 U_i^{1\delta} \tag{84}$$

with $U_i^{1\gamma,\delta}$ defined in (49) and (50).

It is known that generically only simple and double eigenvalues occur in the spectrum of a one-parameter gyroscopic system [30], as we can also see from Fig. 5. In the following, we expand the eigenvalues of the perturbed problem in a series, depending on the Jordan structure of the eigenvalue of the unperturbed problem. The leading terms of these expansions are analytical approximations to the stability boundaries of the system in the parameter space.

3.2.1 Perturbation of Simple Eigenvalues. According to [31], for a simple eigenvalue λ_0 and the corresponding eigenfunction $u(\bar{x})$, we set

$$w(\bar{x})=u(\bar{x})+\varepsilon w_1^\varepsilon(\bar{x})+\cdots \tag{85}$$

$$\lambda=\lambda_0+\varepsilon\lambda_1^\varepsilon+\cdots=\lambda_0+\varepsilon(\gamma_1\lambda_1^\gamma+\delta_1\lambda_1^\delta+\nu_1\lambda_1^\nu)+\cdots \tag{86}$$

Taking into account the dependence of $L(w)$ on λ, we frequently use the notation

$$L(w)=[L(\lambda)](w) \tag{87}$$

Substitution into (81) yields

$$[L(\lambda_0+\varepsilon\lambda_1^\varepsilon+\cdots)+\varepsilon L^{1\varepsilon}(\lambda_0+\varepsilon\lambda_1^\varepsilon+\cdots)](u+\varepsilon w_1^\varepsilon+\cdots)=0 \tag{88}$$

where we can write

$$[L(\lambda_0+\varepsilon\lambda_1^\varepsilon)+\cdots](u)=[L(\lambda_0)](u)+\varepsilon\lambda_1^\varepsilon\left[\frac{\partial L}{\partial\lambda}(\lambda_0)\right](u)+\cdots$$

$$=L(u)+\varepsilon\lambda_1^\varepsilon\frac{\partial L}{\partial\lambda}(u)+\cdots \tag{89}$$

Proceeding similarly with $L^{1\varepsilon}$, we arrive at

$$L(u)+\varepsilon\left[\lambda_1^\varepsilon\frac{\partial L}{\partial\lambda}(u)+L(w_1^\varepsilon)+L^{1\varepsilon}(u)\right]+\cdots=0 \tag{90}$$

Similarly, we get

$$U_i^0(u)+\varepsilon\left[\lambda_1^\varepsilon\frac{\partial U_i^0}{\partial\lambda}(u)+U_i^0(w_1^\varepsilon)+U_i^{1\varepsilon}(u)\right]+\cdots=0 \quad (91)$$

Setting equal to zero the terms of same powers of ε and multiplying with the eigenfunction of the unperturbed adjoint problem v, we obtain

$$\lambda_1^\varepsilon\left\langle\frac{\partial L}{\partial\lambda}(u),v\right\rangle+\langle L(w_1^\varepsilon),v\rangle+\langle L^{1\varepsilon}(u),v\rangle=0 \quad (92)$$

Using $\langle L(w_1^\varepsilon),v\rangle=\langle w_1^\varepsilon,L^*(v)\rangle+\Sigma_{i=1}^n U_i^0(w_1^\varepsilon)V_{n+1-i}^0(v)$, where $n=16$ for our particular problem, and

$$U_i^0(w_1^\varepsilon)=-\lambda_1^\varepsilon\frac{\partial U_i^0(u)}{\partial\lambda}-U_i^{1\varepsilon}(u) \quad (93)$$

we obtain a formula for λ_1^ε, first derived by Kirillov and Seyranian in [23–25] for general two-point non-self-adjoint boundary value problems smoothly depending on the spectral parameter and a vector of physical parameters:

$$\lambda_1^\varepsilon=-\frac{\langle L^{1\varepsilon}(u),v\rangle-\sum_{i=1}^n U_i^{1\varepsilon}(u)V_{n+1-i}^0(v)}{\left\langle\frac{\partial L}{\partial\lambda}(u),v\right\rangle-\sum_{i=1}^n\frac{\partial U_i^0}{\partial\lambda}(u)V_{n+1-i}^0(v)} \quad (94)$$

Substituting $L^{1\varepsilon}=\gamma_1L^{1\gamma}+\delta_1L^{1\delta}$ and $U_i^{1\varepsilon}=\gamma_1U_i^{1\gamma}+\delta_1U_i^{1\delta}$, using (49), (50), and (53)–(61), and taking into account that the eigenfunctions and their first derivatives are continuous (i.e., for example $u(\bar{a}^+)=u(\bar{a}^-)=u(\bar{a})$), we calculate λ_1^γ, λ_1^δ, and λ_1^ν, which for our problem, read

$$\lambda_1^\gamma=\frac{-2\bar{k}u(\bar{a})\bar{v}(\bar{a})+(\bar{h}\bar{k}\mu u(\bar{a})-\bar{h}\bar{N}_0\mu^2u'(\bar{a}))\bar{v}'(\bar{a})}{\int_0^1(2\lambda_0u+2\tilde{\rho}u')\bar{v}\mathrm{d}\bar{x}} \quad (95)$$

$$\lambda_1^\delta=\frac{-2\bar{d}\lambda_0u(\bar{a})\bar{v}(\bar{a})+\bar{h}\bar{d}\lambda_0\mu u(\bar{a})\bar{v}'(\bar{a})}{\int_0^1(2\lambda_0u+2\tilde{\rho}u')\bar{v}\mathrm{d}\bar{x}} \quad (96)$$

and

$$\lambda_1^\nu=-\frac{\int_0^1(2\lambda_0u'+2\tilde{\rho}u'')\bar{v}\mathrm{d}\bar{x}}{\int_0^1(2\lambda_0u+2\tilde{\rho}u')\bar{v}\mathrm{d}\bar{x}} \quad (97)$$

Note that (94) can be regarded as the extension of the formulas derived in [23–25] to the important case of problems containing intermediate boundary conditions.

3.2.2 Perturbation of Double Eigenvalues. Following [31] for double eigenvalues, we set

$$w(\bar{x})=u(\bar{x})+\varepsilon^{\frac{1}{2}}w_1^\varepsilon(\bar{x})+\cdots \quad (98)$$

$$\lambda=\lambda_0+\varepsilon^{\frac{1}{2}}\lambda_1^\varepsilon+\cdots \quad (99)$$

Expanding all the terms in powers of $\varepsilon^{\frac{1}{2}}$ and ordering yields

$$\varepsilon^0:\quad L(u)=0 \quad (100)$$

$$\varepsilon^{\frac{1}{2}}:\quad \lambda_1^\varepsilon\frac{\partial L}{\partial\lambda}(u)+L(w_1^\varepsilon)=0 \quad (101)$$

$$\varepsilon:\quad (\lambda_1^\varepsilon)^2\frac{1}{2}\frac{\partial^2L}{\partial\lambda^2}(u)+\lambda_2^\varepsilon\frac{\partial L}{\partial\lambda}(u)+\lambda_1^\varepsilon\frac{\partial L}{\partial\lambda}(w_1^\varepsilon)+L(w_2^\varepsilon)+L^{1\varepsilon}(u)=0 \quad (102)$$

and similar expressions hold for the U_i, namely,

$$\varepsilon^0:\quad U_i^0(u)=0 \quad (103)$$

$$\varepsilon^{\frac{1}{2}}:\quad \lambda_1^\varepsilon\frac{\partial U_i^0}{\partial\lambda}(u)+U_i^0(w_1^\varepsilon)=0 \quad (104)$$

$$\varepsilon:\quad (\lambda_1^\varepsilon)^2\frac{1}{2}\frac{\partial^2U_i^0}{\partial\lambda^2}(u)+\lambda_2^\varepsilon\frac{\partial U_i^0}{\partial\lambda}(u)+\lambda_1^\varepsilon\frac{\partial U_i^0}{\partial\lambda}(w_1^\varepsilon)+U_i^0(w_2^\varepsilon)+U_i^{1\varepsilon}(u)=0 \quad (105)$$

From (101) follows $w_1^\varepsilon=\lambda_1^\varepsilon u_1+Cu$, where C is a constant. Multiplying the Jordan chain by v, we get

$$\begin{aligned}\langle v,L(w_1^\varepsilon)\rangle+\lambda_1^\varepsilon\left\langle v,\frac{\partial L}{\partial\lambda}(u)\right\rangle&=\sum_{i=1}^n U_i^0(w_1^\varepsilon)V_{n+1-i}^0(v)+\lambda_1^\varepsilon\left\langle v,\frac{\partial L}{\partial\lambda}(u)\right\rangle\\&=-\lambda_1^\varepsilon\sum_{i=1}^n\frac{\partial^2U_i^0}{\partial\lambda^2}(u)V_{n+1-i}^0(v)\\&\quad+\lambda_1^\varepsilon\left\langle v,\frac{\partial L}{\partial\lambda}(u)\right\rangle=0\end{aligned} \quad (106)$$

making use of (104). Multiplication of (102) by v and integration by parts using (105) yields

$$\begin{aligned}\langle v,L(w_2^\varepsilon)\rangle&=\sum_{i=1}^n U_i^0(w_2^\varepsilon)V_{n+1-i}^0(v)=-\sum_{i=1}^n\left[(\lambda_1^\varepsilon)^2\frac{1}{2}\frac{\partial^2U_i^0}{\partial\lambda^2}(u)\right.\\&\quad\left.+\lambda_2^\varepsilon\frac{\partial U_i^0}{\partial\lambda}(u)+\lambda_1^\varepsilon\frac{\partial U_i^0}{\partial\lambda}(w_1)+U_i^{1\varepsilon}(u)\right]V_{n+1-i}^0(v)\\&=-(\lambda_1^\varepsilon)^2\frac{1}{2}\left\langle v,\frac{\partial^2L}{\partial\lambda^2}(u)\right\rangle-\lambda_2^\varepsilon\left\langle\frac{\partial L}{\partial\lambda}(u),v\right\rangle\\&\quad-\lambda_1^\varepsilon\left\langle\frac{\partial L}{\partial\lambda}(w_1^\varepsilon),v\right\rangle-\langle L^{1\varepsilon}(u),v\rangle\end{aligned}$$

With $w_1^\varepsilon=\lambda_1^\varepsilon u_1+Cu$ and using (106), this simplifies to the formula

$$(\lambda_1^\varepsilon)^2=-\frac{1}{\sigma_2}\left[\langle L^{1\varepsilon}(u),v\rangle-\sum_{i=1}^n U_i^{1\varepsilon}(u)V_{n+1-i}^0(v)\right] \quad (107)$$

$$\sigma_2=\sum_{r=1}^2\frac{1}{r!}\left[\left\langle\frac{\partial^rL}{\partial\lambda^r}(u_{2-r}),v\right\rangle-\sum_{i=1}^n\frac{\partial^rU_i^0}{\partial\lambda^r}(u_{2-r})V_{n+1-i,}^0(v)\right] \quad (108)$$

already derived in [23–25]. This is again the extension allowing for intermediate boundary conditions, which can obviously be done for Jordan chains of arbitrary length and several intermediate transition conditions. Using (49), (50), and (53)–(61) for our problem, it reads

$$
\begin{aligned}
(\lambda_1^{\varepsilon})^2 = {} & \gamma_1 \frac{-2\bar{k}u(\bar{a})\bar{v}(\bar{a}) + [\bar{h}\bar{k}\mu u(\bar{a}) - \bar{h}\bar{N}_0\mu^2 u'(\bar{a})]\bar{v}'(\bar{a})}{\displaystyle\int_0^1 (2\lambda_0 u_1 + 2\tilde{\rho}u_1')\bar{v}\,\mathrm{d}\bar{x} + \int_0^1 2u\bar{v}\,\mathrm{d}\bar{x}} \\
& + \delta_1 \frac{-2\bar{d}\lambda_0 u(\bar{a})\bar{v}(\bar{a}) + \bar{h}\bar{d}\lambda_0\mu u(\bar{a})\bar{v}'(\bar{a})}{\displaystyle\int_0^1 (2\lambda_0 u_1 + 2\tilde{\rho}u_1')\bar{v}\,\mathrm{d}\bar{x} + \int_0^1 2u\bar{v}\,\mathrm{d}\bar{x}} \\
& + \nu_1 \frac{\displaystyle -\int_0^1 (2\lambda_0 u' + 2\tilde{\rho}u'')\bar{v}\,\mathrm{d}\bar{x}}{\displaystyle\int_0^1 (2\lambda_0 u_1 + 2\tilde{\rho}u_1')\bar{v}\,\mathrm{d}\bar{x} + \int_0^1 2u\bar{v}\,\mathrm{d}\bar{x}}
\end{aligned}
\qquad (109)
$$

3.3 Stability Boundaries. In Secs. 3.2.1 and 3.2.2, we derived formulas for the change of simple and double eigenvalues occurring in the spectrum of the unperturbed system, caused by small perturbations of the parameters. For a fixed velocity $\bar{\rho}$, the stability region of the system in the parameter plane γ, δ is given by those areas where all eigenvalues have a negative real part. For each simple purely imaginary eigenvalue λ_j of the unperturbed problem, there is a stable region, which in the first approximation, is a half-plane

$$\gamma\,\mathrm{Re}(\lambda_1^{j\gamma}) + \delta\,\mathrm{Re}(\lambda_1^{j\delta}) \leq 0 \quad \forall j \qquad (110)$$

If, at a certain speed $\bar{\rho}$, all eigenvalues are simple, then first approximation to the stable region is given by the intersection of the half-planes (see also [17], [23–25,32]). Depending on the parameters, the intersection can be a sector limited by an angle, a line (for $\mathrm{Re}(\lambda_1^{1\delta})/\mathrm{Re}(\lambda_1^{1\gamma}) = \cdots = \mathrm{Re}(\lambda_1^{n\delta})/\mathrm{Re}(\lambda_1^{n\gamma})$) or a point.

The necessary and sufficient conditions for a double purely imaginary eigenvalue λ_0 not to move to the right-hand side of the complex plane in the first approximation, i.e.,

$$\mathrm{Re}\left\{\sqrt{\mathrm{Re}[(\lambda_1^{\varepsilon})^2] + i\,\mathrm{Im}[(\lambda_1^{\varepsilon})^2]}\right\} \leq 0 \qquad (111)$$

are $\mathrm{Im}[(\lambda_1^{\varepsilon})^2]=0$ and $\mathrm{Re}[(\lambda_1^{\varepsilon})^2]<0$. The condition $\mathrm{Im}[(\lambda_1^{\varepsilon})^2]=0$ defines the line

$$\gamma = \frac{\mathrm{Im}[2\bar{d}\lambda_0 u(\bar{a})\bar{v}(\bar{a}) - \bar{h}\bar{d}\lambda_0\mu u(\bar{a})\bar{v}'(\bar{a})]}{\mathrm{Im}\{2\bar{k}u(\bar{a})\bar{v}(\bar{a}) - [\bar{h}\bar{k}\mu u(\bar{a}) - \bar{h}\bar{N}_0\mu^2 u'(\bar{a})]\bar{v}'(\bar{a})\}}\delta \qquad (112)$$

only half of which can be in the stable region. We are now in the position to draw pictures of the onset of the stability regions.

In Figs. 6 and 7, we see the onset of the stability boundary corresponding to the smallest two pairs of complex conjugate eigenvalues, plotted for the values of parameters

$$\mu = 0.3 \quad \bar{d} = 0.5 \quad \bar{k} = 1 \quad \bar{N}_0 = 0.1 \quad \bar{\kappa} = 3\pi \quad \bar{h} = 0.01 \quad \bar{a} = 0.51 \qquad (113)$$

Below the first critical speed $\tilde{\rho}_1 = 6.99$, we have a simple spectrum with purely imaginary eigenvalues. Therefore, we get a region of asymptotic stability, which for small γ and δ, is approximately a sector limited by an angle.

Therefore, the simultaneous actions of dissipative and nonconservative positional forces can cause both asymptotic stabilization and flutter instability. This is very important, since it occurs in the subcritical range in squeal problems. However, in the subcritical range, in accordance with Fig. 6, it is possible to assign a destabilizing effect to nonconservative forces, as discussed in Sec. 2.4 based on [1,16,19], and a stabilizing effect to damping forces (at least if they cause a positive semidefinite damping operator). This is no longer true in the supercritical range, since the stiffness operator becomes negative definite. Due to the angle singularity

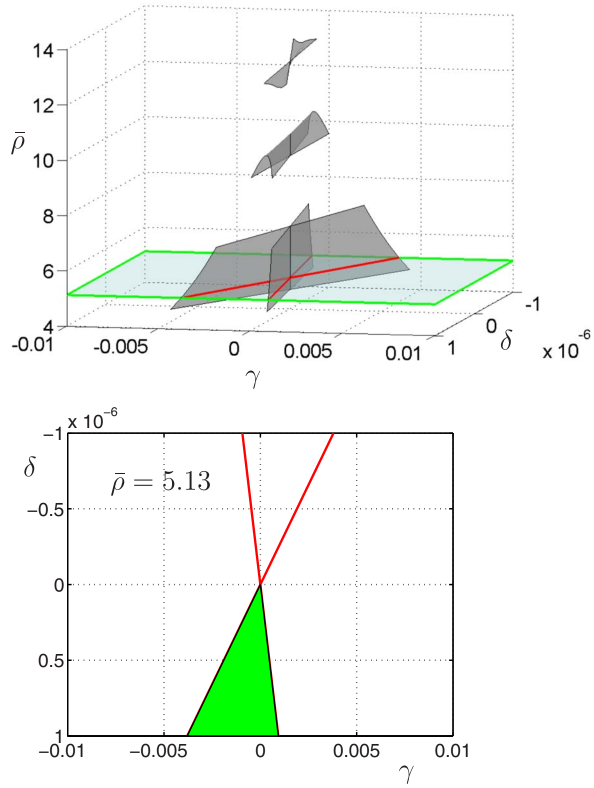


Fig. 6 Stability boundaries (subcritical range)

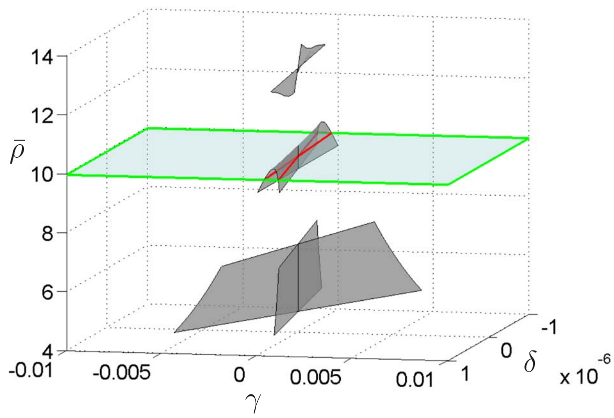
on the stability boundary, the choice of the stabilizing combination of the forces is nontrivial, especially in this case in agreement with [17].

At the critical speed, we get a double zero eigenvalue with a Jordan chain, characterizing the divergence boundary of the unperturbed system. At the second critical speed $\tilde{\rho}_2 = 9.50$, the system stabilizes again. For the perturbed problem, the stability region is again given by a sector limited by an angle.

The next interesting point in the spectrum occurs where the first and second eigenvalues meet with nonvanishing imaginary part ($\tilde{\rho}_3 = 10.30$). Here, we have again a double eigenvalue with a Jordan chain. Towards this point, the sector of the stable region shrinks to a line since the stability boundaries of first and second eigenvalue coincide at this point, as can be seen from the perturbation formulas derived for the simple and the double eigenvalue in Secs. 3.2.1 and 3.2.2. What we see around this point in Fig. 7 is, in fact, a generic singularity of the stability boundary of a three-parameter system ($\bar{\rho}$, γ, δ). These singularities have been investigated by Arnold [30] and the one that we are observing is the Whitney umbrella caused by a double purely imaginary eigenvalue with a Jordan block. Above the speed corresponding to the Whitney umbrella, the unperturbed system suffers flutter instability. Increasing the speed of the beam, the unperturbed system stabilizes again at a critical speed $\tilde{\rho}_4 = 12.78$, corresponding to a double eigenvalue with a Jordan block, again yielding a Whitney umbrella. Afterwards, the system again loses stability by divergence, and additionally suffers flutter at a later stage.

The Whitney umbrella also appears in other problems, e.g., on the stability boundaries of the Beck column with external and internal damping (see [23,25]). It was also found in general two degree of freedom linear gyroscopic systems with damping and circulatory forces considered in [17], as well as in circulatory systems with small velocity dependent forces [33].

We now give an expression for the Whitney umbrella. Suppose the beam without pads, i.e., $\gamma = \delta = 0$, is moving with the speed $\bar{\rho}$



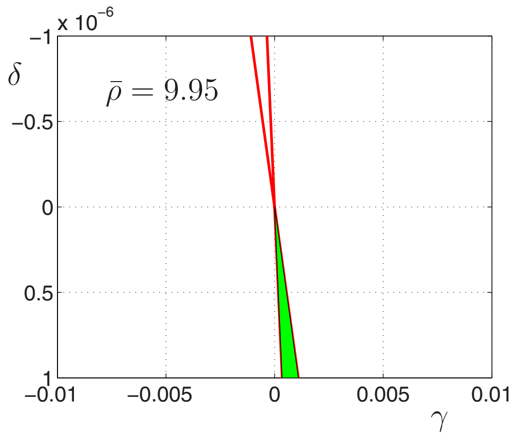


Fig. 7 Stability boundaries (supercritical range)

$=\tilde{\rho}$ corresponding to the first double purely imaginary eigenvalue $\lambda_0=\tilde{\lambda}_0$, the corresponding eigenfunction $u=\tilde{u}$, and associated function $u_1=\tilde{u}_1$. Now the velocity is changed slightly by the small parameter ν. Formula (109) yields

$$(\lambda_1^{\nu})^2=(\lambda_1^{\varepsilon})^2=-\nu_1\frac{\langle\tilde{\lambda}_0\tilde{u}'+\tilde{\rho}\tilde{u}'',\tilde{u}\rangle}{\langle\tilde{\lambda}_0\tilde{u}_1+\tilde{\rho}\tilde{u}_1+\tilde{u},\tilde{u}\rangle} \tag{114}$$

that is seen to be a real quantity, because $\langle u_1,L(u_1)\rangle=-\langle u_1,(\partial L/\partial\lambda)(u)\rangle$ is real, which follows from the defining equations of the Jordan chain and integration by parts. Consequently, in the vicinity of the first flutter boundary $\bar{\rho}=\tilde{\rho}+\nu=\tilde{\rho}+\varepsilon\nu_1$, the increment $\sqrt{\nu}\lambda_1^{\nu}$ to the unperturbed double eigenvalue $\tilde{\lambda}_0$ is purely imaginary for $\nu<0$ and real otherwise. For negative ν, the eigenvalue $\tilde{\lambda}_0$ splits into two simple purely imaginary eigenvalues $\pm\bar{\lambda}_0$.

Perturbation of the system for arbitrary $\bar{\rho}$ corresponding to a simple spectrum of the unperturbed problem with the forces coming from the pads yielded $\bar{\lambda}(\bar{\rho})=\bar{\lambda}_0(\bar{\rho})+\gamma\lambda_1^{\gamma}(\bar{\rho})+\delta\lambda_1^{\delta}(\bar{\rho})$ for the eigenvalues meeting at $\tilde{\rho}$. The coefficients $\lambda_1^{\gamma}(\bar{\rho})$ and $\lambda_1^{\delta}(\bar{\rho})$ were given in (95) and (96), depending on $\bar{\rho}$ through the eigenvalues and eigenfunctions of the unperturbed problem ($\delta=\gamma=0$). Substituting $\bar{\lambda}_0=\tilde{\lambda}_0\pm\sqrt{\nu}\lambda_1^{\nu}+\cdots$ and $\bar{u}=\tilde{u}\pm\sqrt{\varepsilon}\lambda_1^{\nu}\tilde{u}_1$ into (95) and (96) and postulating that $\bar{\lambda}(\bar{\rho})=0$, we obtain an approximate equation for the critical speed for the flutter boundary for the beam with pads of the form

$$\bar{\rho}=\tilde{\rho}+\left(\frac{A\delta+B\gamma}{C\delta+D\gamma}\right)^2 \tag{115}$$

where A,B,C, and D are constants depending only on the spectral data of the unperturbed problem at $\bar{\rho}=\tilde{\rho}$, $\delta=\gamma=0$, and corresponding to the double eigenvalue $\tilde{\lambda}_0$. Equation (115) is of the canonical form for the Whitney umbrella; see [17,30].

4 Conclusion

In this paper we considered a moving beam with clamped boundary conditions in frictional contact with idealized pads. The system's equations of motion were derived, and the interactions between beam and rod equations were identified. Due to the pads, self-excited vibrations can arise in the system originating from instabilities of the trivial solution of the beam equation. The investigated mechanism not only occurs in beams, but can also be observed in rotating disks, providing an explanation for the phenomenon of squeal, and probably also in other moving continua, like shells. The problem was investigated using a perturbation approach, which enabled us to calculate analytic approximations to the stability boundaries. It was found that on the stability boundary of the system, there are generic singularities corresponding to double eigenvalues with a Jordan chain, and analytic approximations for the splitting of eigenvalues in the vicinity of these singularities were calculated. The model is an example for a system losing Hamiltonian symmetry only due to perturbations in the boundary conditions. Insights gained from the problem carry over to other problems with moving media, and are to be investigated in future research.

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