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Instabilities induced by dissipation

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Abstract: The paradox of destabilization of a conservative or non-conservative system by small dissipation, or Ziegler's paradox (1952), has stimulated a growing interest in the sensitivity of reversible and Hamiltonian systems with respect to dissipative perturbations. Since the last decade it has been widely accepted that dissipation-induced instabilities are closely related to singularities arising on the stability boundary, associated with Whitney's umbrella. The first explanation of Ziegler's paradox was given (much earlier) by Oene Bottema in 1956. We discuss aspects of the mechanics and geometry of dissipation-induced instabilities with an application to rotor dynamics

1. Introduction

There is a fascinating category of mechanical and physical systems which exhibit the following paradoxical behavior: when modeled as systems without damping they possess stable equilibria or stable steady motions, but when small damping is introduced, some of these equilibria or steady motions become unstable. A systematic survey of the literature is given by Kirillov and Verhulst (2010).

The paradoxical effect of damping on dynamic instability was noticed first for rotor systems which have stable steady motions for a certain range of speed, but which become unstable when the speed is changed to a value outside the range. In 1924, Kimball studied the destabilization of a flexible rotor in stable rotation at a speed above the critical speed for resonance. In fact, in 1879 Thomson and Tait showed already that a statically unstable conservative system which has been stabilized by gyroscopic forces could be destabilized again by the introduction of small damping forces. However, the destabilization by damping, using Routh's theorems, is implicit in their calculations, it is not formulated as a paradox.

Ziegler's paradox

In 1952 Hans Ziegler of ETH Zurich published a paper that became classical and widely known in the community of mechanical engineers; it also attracted the attention of mathematicians. Ziegler was interested in flutter problems in aerodynamics and considered a double pendulum, fixed at one end and compressed by a tangential end load. He unexpectedly encountered a phenomenon with a paradoxal character: the domain of stability of the Ziegler's pendulum changes in a discontinuous way when one passes from the case in which the damping is very small to that where it has vanished, see Ziegler (1952, 1953).

In the conclusion to his classical book, Bolotin (1961) emphasized that the discrepancy between the stability domains of undamped non-conservative systems and that of systems with infinitesimally small dissipation is a topic of the greatest theoretical interest in stability theory. Encouraging further research of the destabilization paradox, Bolotin was not aware that the crucial ideas for its explanation were formulated by Bottema as early as 1956. Surprisingly, this paper surpassed the attention of most scientists during five decades.

2. Bottema opened Whitney's umbrella

In a remarkable paper of 1943, Hassler Whitney described singularities of maps from $\mathbb{R}^n$ into $\mathbb{R}^m$ with $m = 2n - 1$. It turns out that in this case a special kind of singularity plays a prominent role. Later, the local geometric structure of the manifold near the singularity has been aptly called 'Whitney's umbrella'. In Fig. 1 we reproduce the original sketch of the singular surface from the companion article (Whitney, 1944).

The basic idea is this. Consider a n-dimensional manifold with a singularity at the origin. The manifold is mapped into m-space with $m = 2n - 1$. To be concrete, assume $n = 2$, $m = 3$, the simplest interesting case. In a neighborhood of the origin it is possible to find coordinates such that we have exactly

$$y_1 = x_1^2,\ y_2 = x_2,\ y_3 = x_1 x_2, \tag{1}$$

so that $y_1 \geq 0$ and on eliminating x_1 and x_2:

$$y_1 y_2^2 - y_3^2 = 0.$$

Starting on the y_2-axis for $y_1 = y_3 = 0$, the surface widens up for increasing values of y_1. For each y_2, the cross-section is a parabola; as y_2 passes through 0, the parabola degenerates to a half-ray, and opens out again (with sense reversed); see Fig. 1.

The analysis of singularities of functions and maps has become a fundamental ingredient for bifurcation studies of differential equations. After the pioneering work of Hassler Whitney

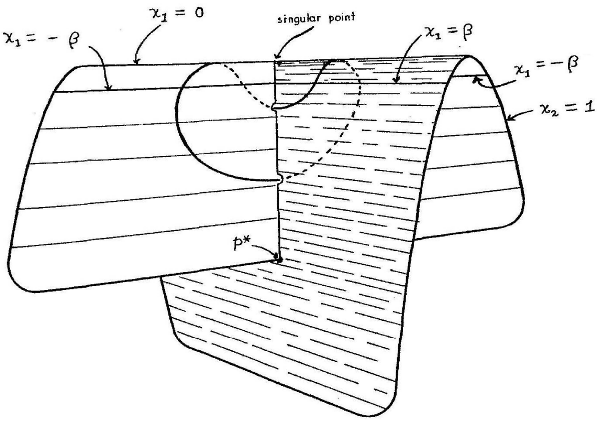

Figure 1: Whitney's original 1944 sketch of the umbrella.

and Marston Morse, it has become a huge research field, both in theoretical investigations and in applications. In 1943 it was hard to imagine that this study of global analysis, a pure mathematical abstraction, would find already an industrial application in the next decade.

Bottema's solution

In 1956, there appeared an article by Oene Bottema (1901-1992), that outstripped later findings for decades. Bottema's work in 1955 can be seen as an introduction, it was directly motivated by Ziegler's paradox. In 1956 he considers a much more general class of small oscillations of non-conservative systems near the equilibrium configuration $x = y = 0$:

$$\begin{aligned} \ddot{x} + a_{11}x + a_{12}y + b_{11}\dot{x} + b_{12}\dot{y} &= 0, \\ \ddot{y} + a_{21}x + a_{22}y + b_{21}\dot{x} + b_{22}\dot{y} &= 0, \end{aligned}$$

where a_{ij} and b_{ij} are constants. The characteristic equation for the frequencies of the small oscillations around equilibrium is

$$Q := \lambda^4 + a_1\lambda^3 + a_2\lambda^2 + a_3\lambda + a_4 = 0,$$

where the parameters $a_1, \cdots, a_4$ depend on the 8 parameters of the system. For the equilibrium to be stable all roots of the characteristic equation have to be semi-simple and have real parts which are non-positive, in the case of multiple eigenvalues the real parts have to be negative. After a long linear algebra analysis, we find that the condition for the surface

V that separates stable and unstable motion is

$$a_1 a_2 a_3 = a_1^2 + a_3^2$$

This is the equation of a surface V of the third degree, which we have to consider for $a_1 \geq 0$, $a_3 \geq 0$. It is equivalent to Whitney's umbrella in the case $n = 2, m = 3$. Note that we started off with 8 parameters in the system, but that the surface V bounding the stability domain is described by 3 parameters.

Interestingly, Bottema's analysis in 1956 also shows that the ratio of the damping coefficients is important. Explicitly, in the case of symmetric damping, $b_{11} = b_{22}$, the destabilization is not very effective. We show this by a simple example. Consider the system

$$\begin{aligned} \ddot{x} + x + y + \kappa_1 \dot{x} &= 0, \\ \ddot{y} - x + \omega^2 y + \kappa_2 \dot{y} &= 0, \end{aligned}$$

with damping coefficients $(\kappa_1, \kappa_2 \geq 0)$. The characteristic equation for the eigenvalues becomes

$$(\lambda^2 + \kappa_1 \lambda + 1)(\lambda^2 + \kappa_2 \lambda + \omega^2) + 1 = 0.$$

Without damping, $\kappa_1 = \kappa_2 = 0$, the trivial solution is unstable if $0 < \omega^2 < 3$ and stable if $\omega^2 > 3$. In the case of stability, the eigenvalues are purely imaginary. If $\omega^2 = 3$ we have a so-called Krein-collision.

We present the eigenvalues without and with damping for $\omega^2 = 4$ using MATLAB. We have without damping

$$\omega^2 = 4, \kappa_1 = \kappa_2 = 0 \text{ eigenvalues} : \pm 1.9021i, \ \pm 1.1756i.$$

A type of asymmetric damping: $\kappa_1 > 0, \kappa_2 = 0$.

$$\begin{aligned} \omega^2 &= 4, \kappa_1 = 0.1 \text{ eigenvalues} : -0.05851 \pm 1.1736i, \ +0.0085 \pm 1.9029i; \\ \omega^2 &= 4, \kappa_1 = 0.2 \text{ eigenvalues} : -0.1164 \pm 1.1678i, \ +1.0164 \pm 1.9053i. \end{aligned}$$

Damping in the first degree of freedom (x) destabilizes. Now symmetric damping:

$$\begin{aligned} \omega^2 &= 4, \kappa_1 = \kappa_2 = 0.1 \text{ eigenvalues} : -0.0500 \pm 1.9015i, \ -0.0500 \pm 1.1745i; \\ \omega^2 &= 4, \kappa_1 = \kappa_2 = 0.2 \text{ eigenvalues} : -0.1000 \pm 1.8995i, \ -0.1000 \pm 1.1713i. \end{aligned}$$

In the case of symmetric damping we have not necessarily destabilization.

3. Parametric resonance in systems with dissipation.

Parametric resonance arises usually in applications if we have an independent (periodic) source of energy. The classical example is the mathematical pendulum with oscillating support and a typical equation studied in this context is the Mathieu equation. See Fig. 2(a) for this classical case.

In applications with parametric excitation where usually more degrees of freedom play a part, many combination resonances are possible. In what follows, the so-called sum resonance will be important.

Rotor dynamics without damping

The effects of adding linear damping to a parametrically excited system have already been observed and described in for instance Bolotin (1963). The following example is based on Ruijgrok et al. (1993), see also Hoveijn and Ruijgrok (1995).

Consider a rigid rotor consisting of a heavy disk of mass M which is rotating with constant rotation speed Ω around an axis. The axis of rotation is elastically mounted on a foundation; the connections which are holding the rotor in an upright position are also elastic. Assuming small oscillations in the upright position, frequency 2η, the equations of motion without damping become after rescaling:

$$\begin{aligned}
\ddot{x} + 2\alpha\dot{y} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)x &= 0, \\
\ddot{y} - 2\alpha\dot{x} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)y &= 0.
\end{aligned} \tag{2}$$

The parameter α is proportional to the rotation speed Ω. System (2) constitutes a conservative system of coupled Mathieu-like equations. The natural frequencies of the unperturbed system (2), $\varepsilon = 0$, are $\omega_1 = \sqrt{\alpha^2 + 1} + \alpha$ and $\omega_2 = \sqrt{\alpha^2 + 1} - \alpha$. It is shown in Ruijgrok et al. (1993), using complex variables, that we can transform this system to two identical Mathieu equations.

Using the classical and well-known results for the Mathieu equation, we conclude that the trivial solution is stable for ε small enough, provided that $\sqrt{1 + \alpha^2}$ is not close to $n\eta$, for $n = 1, 2, 3, \ldots$. The first-order and most prominent interval of instability, $n = 1$, arises if $\sqrt{1 + \alpha^2} \approx \eta$. Note that this instability arises when:

$$\omega_1 + \omega_2 = 2\eta,$$

i.e. when the sum of the eigenfrequencies of the unperturbed system equals the excitation frequency 2η which is the sum resonance of first order. The domain of instability is bounded

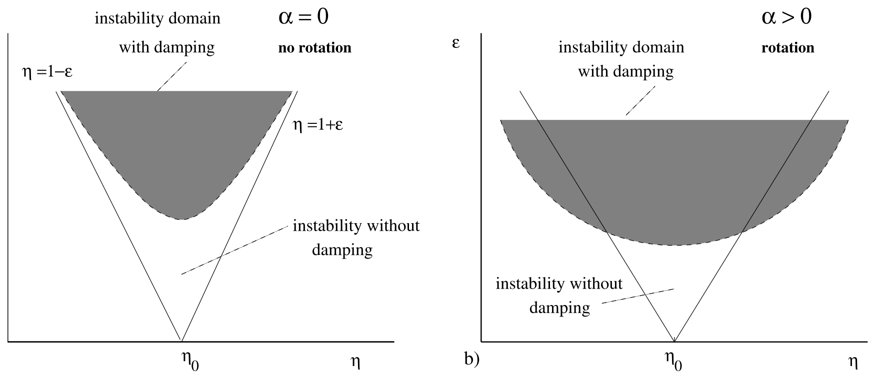

Figure 2: (a) The classical case as we find for instance for the Mathieu equation with and without damping; in the case of damping the instability tongue is lifted off from the η-axis and the instability domain is reduced. (b) The instability tongues for the rotor system. Again, because of damping the instability tongue is lifted off from the η-axis, but the tongue broadens. The boundaries of the V-shaped tongue without damping are to first approximation described by the expression $\eta = \sqrt{1+\alpha^2}(1 \pm \varepsilon)$, $\eta_0 = \sqrt{1+\alpha^2}$.

by:

$$\eta_b = \sqrt{1+\alpha^2}\ (1 \pm \varepsilon) + O(\varepsilon^2) \quad . \tag{3}$$

See Fig. 2(b) where the V-shaped instability domain is presented in the case of rotor rotation ($\alpha \neq 0$) without damping. Higher order combination resonances can be studied in the same way; the domains of instability in parameter space continue to narrow as n increases.

Rotor dynamics with damping

We add small linear damping to system (2), with positive damping parameter $\mu = 2\varepsilon\kappa$. This leads to the equations:

$$\begin{aligned} \ddot{x} + 2\alpha\dot{y} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)x + 2\varepsilon\kappa\dot{x} &= 0, \\ \ddot{y} - 2\alpha\dot{x} + (1 + 4\varepsilon\eta^2 \cos 2\eta t)y + 2\varepsilon\kappa\dot{y} &= 0. \end{aligned} \tag{4}$$

Because of the damping term, we can no longer reduce the system to two identical second order real equations, as we did previously.

To calculate the instability interval around the value $\eta_0 = \frac{1}{2}(\omega_1 + \omega_2) = \sqrt{\alpha^2+1}$, we apply normal form or (periodic solution) perturbation theory, see Ruijgrok et al. (1993) for details, to find for the stability boundary:

$$\eta_b = \sqrt{1+\alpha^2}\left(1 \pm \sqrt{(1+\alpha^2)\varepsilon^2 - \quad {\frac{\mu}{2\eta_0}}^{2}} + ...\right) \quad . \tag{5}$$

It follows that, as in other examples we have seen, the domain of instability actually becomes *larger* when damping is introduced. See Fig. 2b.

The instability interval, shows a discontinuity at $\kappa = 0$. If μ or $\kappa \to 0$, then the boundaries of the instability domain tend to the limits $\eta_b \to \sqrt{1+\alpha^2}(1 \pm \varepsilon\sqrt{1+\alpha^2})$ which differs from the result we found when $\kappa = 0:\ \eta_b = \sqrt{1+\alpha^2}(1 \pm \varepsilon)$. For reasons of comparison, we display the instability tongues in Fig. 2 in the four cases with and without rotation, with and without damping.

Mathematically, the bifurcational behavior is again described by the Whitney umbrella as indicated before. In mechanical terms, the broadening of the instability-domain is caused by the coupling between the two degrees of freedom of the rotor in lateral directions which arises in the presence of damping.

4. Conclusions

- It is remarkable that Bottema's solution in 1956 was ignored for such a long time. For instance Google Scholar gives no citations of the paper in the period 1956-2008.
- The generality of the results described in section 2, enable us to discuss the part played by symmetric damping. One should consult the original 1956 paper to observe the behavior of the eigenvalues with regards to the damping coefficients.
- In the context of dissipation-induced instability, the influence of asymmetric and symmetric damping was studied extensively by Kirillov (2005b, 2007), Kirillov and Seyranian (2005a). In these papers Bottema's results were also generalized to higher (more than 4) dimensions. For the rotor system of section 3, the analysis regarding asymmetric damping was carried out by Hoveijn and Ruijgrok (1995).
- Note that the phenomena described here are basically linear and in this sense complete as locally the dynamics is dominated by the linear terms. Further away from equilibrium and in some critical cases, Krein-collision or small real parts near the umbrella surface, nonlinear terms may come into play.

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