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# Eigenvalue bifurcation in multiparameter families of non-self-adjoint operator matrices

O. N. Kirillov

**Abstract.** We consider two-point non-self-adjoint boundary eigenvalue problems for linear matrix differential operators. The coefficient matrices in the differential expressions and the matrix boundary conditions are assumed to depend analytically on the complex spectral parameter $\lambda$ and on the vector of real physical parameters $\mathbf{p}$. We study perturbations of semi-simple multiple eigenvalues as well as perturbations of non-derogatory eigenvalues under small variations of $\mathbf{p}$. Explicit formulae describing the bifurcation of the eigenvalues are derived. Application to the problem of excitation of unstable modes in rotating continua such as spherically symmetric MHD $\alpha^2$-dynamo and circular string demonstrates the efficiency and applicability of the approach.

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**Keywords.** Operator matrix · Non-self-adjoint boundary eigenvalue problem · Keldysh chain · Multiple eigenvalue · Diabolical point · Exceptional point · Perturbation · Bifurcation · Stability · Veering · Spectral mesh · Rotating continua.

## 1. Introduction

Non-self-adjoint boundary eigenvalue problems for matrix differential operators describe distributed non-conservative systems with the coupled modes and appear in structural mechanics, fluid dynamics, magnetohydrodynamics, to name a few.

Practical needs for optimization and rational experiment planning in modern applications allow both the differential expression and the boundary conditions to depend analytically on the spectral parameter and smoothly on several physical parameters (which can be scalar or distributed) [1–3]. According to the ideas going back to von Neumann and Wigner [4], in the multiparameter operator families, eigenvalues with various algebraic and geometric multiplicities can be generic [5,6]. In some applications additional symmetries yield the existence of *spectral meshes* [7] in the plane 'eigenvalue versus parameter' containing infinite number of nodes with multiple eigenvalues, see, e.g, [8–10] and references therein. As it has been pointed out already by Rellich [11] sensitivity analysis of multiple eigenvalues is complicated by their non-differentiability as functions of several parameters. Singularities corresponding to multiple eigenvalues [5] are related to such important effects as destabilization paradox in near-Hamiltonian and near-reversible systems [12–19], geometric phase [20], reversals of the geomagnetic field [21,3], emission of sound by rotating continua interacting with the friction pads [22–24,50] and other phenomena [25].

An increasing number of multiparameter non-self-adjoint boundary eigenvalue problems and the need for simple constructive estimates of critical parameters and eigenvalues as well as for verification of numerical codes, require development of applicable methods, allowing one to track relatively easily and conveniently the changes in simple and multiple eigenvalues and the corresponding eigenvectors due

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to variation of the differential expression and especially due to transition from one type of boundary conditions to another one without discretization of the original distributed problem, see, e.g., [1–3,9].

A systematical study of bifurcation of eigenvalues of a non-self-adjoint linear operator $L_0$ due to perturbation $L_0+\varepsilon L_1$, where $\varepsilon$ is a small parameter, dates back to the 1950s. Apparently, Krein was the first who derived a formula for the splitting of a double eigenvalue with the Jordan block at the Hamiltonian 1 : 1 resonance (*the Krein collision* [26]), which was expressed through the generalized eigenvectors of the double eigenvalue [27]. Vishik and Lyusternik and Lidskii created a perturbation theory for nonsymmetric matrices and non-self-adjoint differential operators allowing one to find the perturbation coefficients of eigenvalues and eigenfunctions in an explicit form by means of the eigenelements of the unperturbed operator [28,29]. Classical monographs by Rellich [30], Kato [31], and Baumgärtel [32], mostly focusing on the self-adjoint case, contain a detailed treatment of eigenvalue problems linearly or quadratically dependent on the spectral parameter.

Multiparameter perturbation theory for simple and multiple eigenvalues of matrices and generalized matrix eigenvalue problems initiated by Sun [33,34] was continued, e.g., in recent works [25,35,36]. Gohberg, Lancaster and Rodman [37], Najman [38], Langer and Najman [39], Hryniv and Lancaster [40], Lancaster, Markus, and Zhou [41], and Kirillov [15] studied perturbation of eigenelements in one- and multiparameter families of analytic matrix functions.

Recently Kirillov and Seyranian derived explicit formulae for bifurcation of multiple eigenvalues and eigenvectors of two-point non-self-adjoint boundary eigenvalue problems with scalar differential expression and boundary conditions, which depend analytically on the spectral parameter and smoothly on a vector of physical parameters, and applied them to the sensitivity analysis of distributed non-conservative problems prone to dissipation-induced instabilities [16,17,42,43]. An extension to the case of intermediate boundary conditions with an application to the problem of the onset of friction-induced oscillations in the moving beam was considered in [22]. In [7] this technique was applied to the study of MHD $\alpha^2$-dynamo operator with idealistic (Dirichlet) boundary conditions.

In the following we develop this approach further and consider boundary eigenvalue problems for linear non-self-adjoint $m$-th order $N \times N$ matrix differential operators on the interval $[0,1] \ni x$. The coefficient matrices in the differential expression and the matrix boundary conditions are assumed to depend analytically on the spectral parameter $\lambda$ and on a vector of real physical parameters $\mathbf{p}$. The matrix formulation of the boundary conditions is chosen for the convenience of its implementation in computer algebra systems for an automatic derivation of the adjoint eigenvalue problem and perturbed eigenvalues and eigenvectors, which is especially helpful when the order of the derivatives in the differential expression is high [44]. Based on the eigenelements of the unperturbed problem explicit formulae are derived describing bifurcation of the semi-simple multiple eigenvalues (diabolical points) as well as non-derogatory eigenvalues (branch points, exceptional points) under small variation of the parameters in the differential expression and in the boundary conditions. Finally, the general technique is applied to the investigation of the onset of oscillatory instability in rotating continua.

## 2. A non-self-adjoint boundary eigenvalue problem for a matrix differential operator

Following [16,17,42,43,45,46] we consider the boundary eigenvalue problem

$$\mathbf{L}(\lambda,\mathbf{p})\mathbf{u}=0, \quad \mathbf{U}_k(\lambda,\mathbf{p})\mathbf{u}=0, \quad k=1,\dots,m, \tag{1}$$

where $\mathbf{u}(x) \in \mathbb{C}^N \otimes C^{(m)}[0,1]$. The differential expression $\mathbf{Lu}$ of the operator is

$$\mathbf{Lu}=\sum_{j=0}^{m}\mathbf{l}_j(x)\partial_x^{m-j}\mathbf{u}, \quad \mathbf{l}_j(x) \in \mathbb{C}^{N\times N} \otimes C^{(m-j)}[0,1], \quad \det[\mathbf{l}_0(x)] \neq 0, \tag{2}$$

and the boundary forms $\mathbf{U}_k\mathbf{u}$ are

$$\mathbf{U}_k\mathbf{u} = \sum_{j=0}^{m-1} \mathbf{A}_{kj}\mathbf{u}_x^{(j)}(x=0) + \sum_{j=0}^{m-1} \mathbf{B}_{kj}\mathbf{u}_x^{(j)}(x=1), \quad \mathbf{A}_{kj}, \mathbf{B}_{kj} \in \mathbb{C}^{N\times N}. \tag{3}$$

Introducing the block matrix $\mathfrak{U} := [\mathfrak{A}, \mathfrak{B}] \in \mathbb{C}^{mN\times 2mN}$ and the vector

$$\mathfrak{u}^T := \left(\mathbf{u}^T(0), \mathbf{u}_x^{(1)T}(0), \ldots, \mathbf{u}_x^{(m-1)T}(0), \mathbf{u}^T(1), \mathbf{u}_x^{(1)T}(1), \ldots, \mathbf{u}_x^{(m-1)T}(1)\right) \in \mathbb{C}^{2mN} \tag{4}$$

the boundary conditions can be compactly rewritten as [16,17]

$$\mathfrak{U}\mathfrak{u} = [\mathfrak{A}, \mathfrak{B}]\mathfrak{u} = 0, \tag{5}$$

where $\mathfrak{A} = (\mathbf{A}_{kj})|_{x=0} \in \mathbb{C}^{mN\times mN}$ and $\mathfrak{B} = (\mathbf{B}_{kj})|_{x=1} \in \mathbb{C}^{mN\times mN}$. It is assumed that the matrices $\mathbf{l}_j$, $\mathfrak{A}$, and $\mathfrak{B}$ are analytic functions of the complex spectral parameter $\lambda$ and of the real vector of physical parameters $\mathbf{p} \in \mathbb{R}^n$. For some fixed vector $\mathbf{p} = \mathbf{p}_0$ the eigenvalue $\lambda_0$, to which the eigenvector $\mathbf{u}_0$ corresponds, is a root of the characteristic equation obtained after substitution of the general solution to equation $\mathbf{Lu} = 0$ into the boundary conditions (5) [45].

Let us introduce a scalar product $\langle \mathbf{u}, \mathbf{v}\rangle := \int_0^1 \mathbf{v}^*\mathbf{u}dx$, where the asterisk denotes complex-conjugate transpose ($\mathbf{v}^* := \overline{\mathbf{v}}^T$) [45]. Taking the scalar product of $\mathbf{Lu}$ and a vector-function $\mathbf{v}$ and integrating it by parts yields the Lagrange formula for the case of operator matrices (cf. [16,17,45,46])

$$\Omega(\mathbf{u}, \mathbf{v}) := \langle \mathbf{Lu}, \mathbf{v}\rangle - \langle \mathbf{u}, \mathbf{L}^\dagger\mathbf{v}\rangle = \mathfrak{v}^*\mathcal{L}\mathfrak{u}, \tag{6}$$

with the adjoint differential expression [45,46]

$$\mathbf{L}^\dagger\mathbf{v} := \sum_{q=0}^{m} (-1)^{m-q}\partial_x^{m-q}\left(\mathbf{l}_q^*\mathbf{v}\right), \tag{7}$$

the vector $\mathfrak{v}$

$$\mathfrak{v}^T := \left(\mathbf{v}^T(0), \mathbf{v}_x^{(1)T}(0), \ldots, \mathbf{v}_x^{(m-1)T}(0), \mathbf{v}^T(1), \mathbf{v}_x^{(1)T}(1), \ldots, \mathbf{v}_x^{(m-1)T}(1)\right) \in \mathbb{C}^{2mN} \tag{8}$$

and the block matrix $\mathcal{L} := (\mathbf{l}_{ij})$

$$\mathcal{L} = \begin{pmatrix} -\mathfrak{L}(0) & 0 \\ 0 & \mathfrak{L}(1) \end{pmatrix}, \quad \mathfrak{L}(x) = \begin{pmatrix} \mathbf{l}_{00} & \mathbf{l}_{01} & \cdots & \mathbf{l}_{0m-2} & \mathbf{l}_{0m-1} \\ \mathbf{l}_{10} & \mathbf{l}_{11} & \cdots & \mathbf{l}_{1m-2} & 0 \\ \vdots & \vdots & \iddots & \vdots & \vdots \\ \mathbf{l}_{m-20} & \mathbf{l}_{m-21} & \cdots & 0 & 0 \\ \mathbf{l}_{m-10} & 0 & \cdots & 0 & 0 \end{pmatrix}, \tag{9}$$

where the matrices $\mathbf{l}_{ij}$ are

$$\mathbf{l}_{ij} := \sum_{k=i}^{m-1-j} (-1)^k M_{ij}^k\, \partial_x^{k-i}\mathbf{l}_{m-1-j-k}, \qquad M_{ij}^k := \begin{cases} \frac{k!}{(k-i)!i!}, & i+j \le m-1 \cap k \ge i \ge 0 \\ 0, & i+j > m-1 \cup k < i. \end{cases} \tag{10}$$

Extend the original matrix $\mathfrak{U}$ (cf. (5)) to a square matrix $\mathcal{U}$, which is made non-degenerate in a neighborhood of the point $\mathbf{p} = \mathbf{p}_0$ and the eigenvalue $\lambda = \lambda_0$ by an appropriate choice of the auxiliary matrices $\widetilde{\mathfrak{A}}(\lambda, \mathbf{p})$ and $\widetilde{\mathfrak{B}}(\lambda, \mathbf{p})$

$$\mathfrak{U} = [\mathfrak{A}, \mathfrak{B}] \hookrightarrow \mathcal{U} := \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \widetilde{\mathfrak{A}} & \widetilde{\mathfrak{B}} \end{pmatrix} \in \mathbb{C}^{2mN\times 2mN}, \quad \widetilde{\mathfrak{U}} := [\widetilde{\mathfrak{A}}, \widetilde{\mathfrak{B}}], \quad \det(\mathcal{U}) \neq 0. \tag{11}$$

For the adjoint boundary conditions $\mathfrak{V}\mathfrak{v} = [\mathfrak{C}, \mathfrak{D}]\mathfrak{v} = 0$ a similar process yields

$$\mathfrak{V} := [\mathfrak{C}, \mathfrak{D}] \hookrightarrow \mathcal{V} := \begin{pmatrix} \mathfrak{C} & \mathfrak{D} \\ \widetilde{\mathfrak{C}} & \widetilde{\mathfrak{D}} \end{pmatrix} \in \mathbb{C}^{2mN\times 2mN}, \quad \widetilde{\mathfrak{V}} := [\widetilde{\mathfrak{C}}, \widetilde{\mathfrak{D}}], \quad \det(\mathcal{V}) \neq 0. \tag{12}$$

Then, the form $\Omega(\mathbf{u}, \mathbf{v})$ in (6) can be represented as [45]

$$\Omega(\mathbf{u}, \mathbf{v}) = (\mathfrak{V}\mathfrak{v})^* \widetilde{\mathfrak{U}}\mathfrak{u} - (\widetilde{\mathfrak{V}}\mathfrak{v})^* \mathfrak{U}\mathfrak{u}, \tag{13}$$

so that without loss in generality we can assume [16,17]

$$\mathcal{L} = \mathfrak{V}^* \widetilde{\mathfrak{U}} - \widetilde{\mathfrak{V}}^* \mathfrak{U}. \tag{14}$$

Hence, we obtain the formula for calculation of the matrix $\mathfrak{V}$ of the adjoint boundary conditions and the auxiliary matrix $\widetilde{\mathfrak{V}}$

$$\begin{bmatrix} -\widetilde{\mathfrak{V}} \\ \mathfrak{V} \end{bmatrix}^* = \mathcal{L}\mathcal{U}^{-1} = \begin{pmatrix} -\mathfrak{L}(0) & 0 \\ 0 & \mathfrak{L}(1) \end{pmatrix} \begin{pmatrix} \mathfrak{A} & \mathfrak{B} \\ \widetilde{\mathfrak{A}} & \widetilde{\mathfrak{B}} \end{pmatrix}^{-1}, \tag{15}$$

which exactly reproduces and extends the corresponding result of [16,17]. Differentiating the equation (14) we find

$$\partial_\lambda^r \mathcal{L} = \sum_{k=0}^{r} \binom{r}{k} \left[ \left(\partial_\lambda^{r-k} \mathfrak{V}\right)^* \partial_\lambda^k \widetilde{\mathfrak{U}} - \left(\partial_\lambda^{r-k} \widetilde{\mathfrak{V}}\right)^* \partial_\lambda^k \mathfrak{U} \right]. \tag{16}$$

## 3. Perturbation of eigenvalues

Assume that in the neighborhood of the point $\mathbf{p} = \mathbf{p}_0$ the spectrum of the boundary eigenvalue problem (1) is discrete. Denote $\mathbf{L}_0 = \mathbf{L}(\lambda_0, \mathbf{p}_0)$ and $\mathfrak{U}_0 = \mathfrak{U}(\lambda_0, \mathbf{p}_0)$. Let us consider an analytic perturbation of parameters in the form $\mathbf{p} = \mathbf{p}(\varepsilon)$ where $\mathbf{p}(0) = \mathbf{p}_0$ and $\varepsilon$ is a small real number. Then, as in the case of analytic matrix functions [15,39–41], the Taylor decomposition of the differential operator matrix $\mathbf{L}(\lambda, \mathbf{p}(\varepsilon))$ and the matrix of the boundary conditions $\mathfrak{U}(\lambda, \mathbf{p}(\varepsilon))$ are [16,17,42,43]

$$\mathbf{L}\left(\lambda, \mathbf{p}(\varepsilon)\right) = \sum_{r,s=0}^{\infty} \frac{(\lambda-\lambda_0)^r}{r!} \varepsilon^s \mathbf{L}_{rs}, \quad \mathfrak{U}(\lambda, \varepsilon) = \sum_{r,s=0}^{\infty} \frac{(\lambda-\lambda_0)^r}{r!} \varepsilon^s \mathfrak{U}_{rs}, \tag{17}$$

with $\mathbf{L}_{00} = \mathbf{L}_0$, $\mathfrak{U}_{00} = \mathfrak{U}_0$, and

$$\mathbf{L}_{r0} = \partial_\lambda^r \mathbf{L}, \quad \mathfrak{U}_{r0} = \partial_\lambda^r \mathfrak{U}; \quad \mathbf{L}_{r1} = \sum_{j=1}^{n} \dot{p}_j\, \partial_\lambda^r \partial_{p_j} \mathbf{L}, \quad \mathfrak{U}_{r1} = \sum_{j=1}^{n} \dot{p}_j\, \partial_\lambda^r \partial_{p_j} \mathfrak{U},$$

where dot denotes differentiation with respect to $\varepsilon$ at $\varepsilon = 0$ and partial derivatives are evaluated at $\mathbf{p} = \mathbf{p}_0$, $\lambda = \lambda_0$. Our aim is to derive explicit expressions for the leading terms in the expansions for multiple-semisimple and non-derogatory eigenvalues and for the corresponding eigenvectors.

### 3.1. Semi-simple eigenvalue

Let at the point $\mathbf{p} = \mathbf{p}_0$ the spectrum contain a semi-simple $\mu$-fold eigenvalue $\lambda_0$ with $\mu$ linearly-independent eigenvectors $\mathbf{u}_0(x)$, $\mathbf{u}_1(x)$, ..., $\mathbf{u}_{\mu-1}(x)$. Then, the perturbed eigenvalue $\lambda(\varepsilon)$ and the eigenvector $\mathbf{u}(\varepsilon)$ are represented as Taylor series in $\varepsilon$ [28,34,40,41,43]

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots, \quad \mathbf{u} = \mathbf{b}_0 + \varepsilon\mathbf{b}_1 + \varepsilon^2\mathbf{b}_2 + \cdots. \tag{18}$$

For the sake of brevity we do not concern here the existence and convergence of expansions (18) that can be treated by the methods described, e.g., in [28].

Substituting expansions (17) and (18) into (1) and collecting the terms with the same powers of $\varepsilon$ we derive the boundary value problems

$$\mathbf{L}_0\mathbf{b}_0 = 0, \quad \mathfrak{U}_0\mathfrak{b}_0 = 0, \tag{19}$$

$$\mathbf{L}_0\mathbf{b}_1 + (\lambda_1\mathbf{L}_{10} + \mathbf{L}_{01})\mathbf{b}_0 = 0, \quad \mathfrak{U}_0\mathfrak{b}_1 + (\lambda_1\mathfrak{U}_{10} + \mathfrak{U}_{01})\mathfrak{b}_0 = 0, \tag{20}$$

The scalar product of (20) with the eigenvectors $\mathbf{v}_j$, $j = 0, 1, \ldots, \mu-1$ of the adjoint boundary eigenvalue problem

$$\mathbf{L}_0^\dagger\mathbf{v} = 0, \quad \mathfrak{V}_0\mathfrak{v} = 0 \tag{21}$$

yields $\mu$ equations

$$\langle\mathbf{L}_0\mathbf{b}_1, \mathbf{v}_j\rangle = -\langle\mathbf{L}_{01}\mathbf{b}_0, \mathbf{v}_j\rangle - \lambda_1\langle\mathbf{L}_{10}\mathbf{b}_0, \mathbf{v}_j\rangle. \tag{22}$$

With the use of the Lagrange formula (6), (14) and the boundary conditions (20) the left hand side of (22) takes the form

$$\langle\mathbf{L}_0\mathbf{b}_1, \mathbf{v}_j\rangle = (\widetilde{\mathfrak{V}}_0\mathfrak{v}_j)^*(\mathfrak{U}_{01}\mathfrak{b}_0 + \lambda_1\mathfrak{U}_{10}\,\mathfrak{b}_0). \tag{23}$$

Together (22) and (23) result in the equations

$$\lambda_1\left(\langle\mathbf{L}_{10}\,\mathbf{b}_0, \mathbf{v}_j\rangle + (\widetilde{\mathfrak{V}}_0\mathfrak{v}_j)^*\mathfrak{U}_{10}\,\mathfrak{b}_0\right) = -\langle\mathbf{L}_{01}\mathbf{b}_0, \mathbf{v}_j\rangle - (\widetilde{\mathfrak{V}}_0\mathfrak{v}_j)^*\mathfrak{U}_{01}\mathfrak{b}_0. \tag{24}$$

Assuming in the equations (24) the vector $\mathbf{b}_0(x)$ as a linear combination

$$\mathbf{b}_0(x) = c_0\mathbf{u}_0(x) + c_1\mathbf{u}_1(x) + \cdots + c_{\mu-1}\mathbf{u}_{\mu-1}(x), \tag{25}$$

and taking into account that

$$\mathfrak{b}_0 = c_0\mathfrak{u}_0 + c_1\mathfrak{u}_1 + \cdots + c_{\mu-1}\mathfrak{u}_{\mu-1}, \tag{26}$$

where $\mathbf{c}^T = (c_0, c_1, \ldots, c_{\mu-1})$, we arrive at the matrix eigenvalue problem (cf. [41])

$$-\mathbf{F}\mathbf{c} = \lambda_1\mathbf{G}\mathbf{c}. \tag{27}$$

The entries of the $\mu \times \mu$ matrices $\mathbf{F}$ and $\mathbf{G}$ are defined by the expressions

$$F_{ij} = \langle\mathbf{L}_{01}\mathbf{u}_j, \mathbf{v}_i\rangle + \mathfrak{v}_i^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_{01}\mathfrak{u}_j, \quad G_{ij} = \langle\mathbf{L}_{10}\mathbf{u}_j, \mathbf{v}_i\rangle + \mathfrak{v}_i^*\widetilde{\mathfrak{V}}_0^*\,\mathfrak{U}_{10}\,\mathfrak{u}_j. \tag{28}$$

Therefore, in the first approximation the splitting of the semi-simple eigenvalue due to variation of parameters $\mathbf{p}(\varepsilon)$ is $\lambda = \lambda_0 + \varepsilon\lambda_1 + o(\varepsilon)$, where the coefficients $\lambda_1$ are generically $\mu$ distinct roots of the $\mu$-th order polynomial

$$\det(\mathbf{F} + \lambda_1\mathbf{G}) = 0. \tag{29}$$

For $\mu = 1$ the formulas (28) and (29) describe perturbation of a simple eigenvalue

$$\lambda = \lambda_0 - \varepsilon\frac{\langle\mathbf{L}_{01}\mathbf{u}_0, \mathbf{v}_0\rangle + \mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_{01}\mathfrak{u}_0}{\langle\mathbf{L}_{10}\mathbf{u}_0, \mathbf{v}_0\rangle + \mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\,\mathfrak{U}_{10}\,\mathfrak{u}_0} + o(\varepsilon). \tag{30}$$

The formulas (28), (29), and (30) generalize the corresponding results of the works [34,41,43] to the case of the multiparameter non-self-adjoint boundary eigenvalue problems for operator matrices.

### 3.2. Non-derogatory eigenvalue

Let at the point $\mathbf{p} = \mathbf{p}_0$ the spectrum contain a $\mu$-fold eigenvalue $\lambda_0$ with the Keldysh chain of length $\mu$, consisting of the eigenvector $\mathbf{u}_0(x)$ and the associated vectors $\mathbf{u}_1(x), \dots, \mathbf{u}_{\mu-1}(x)$ that solve the boundary value problems [45,46]

$$\mathbf{L}_0\mathbf{u}_0 = 0, \quad \mathfrak{U}_0\mathfrak{u}_0 = 0, \tag{31}$$

$$\mathbf{L}_0\mathbf{u}_j = -\sum_{r=1}^{j}\frac{1}{r!}\partial_\lambda^r\mathbf{L}\mathbf{u}_{j-r}, \quad \mathfrak{U}_0\mathfrak{u}_j = -\sum_{r=1}^{j}\frac{1}{r!}\partial_\lambda^r\mathfrak{U}\mathfrak{u}_{j-r}. \tag{32}$$

Consider vector-functions $\mathbf{v}_0(x), \mathbf{v}_1(x), \dots, \mathbf{v}_{\mu-1}(x)$. Let us take scalar product of the differential equation (31) and the vector-function $\mathbf{v}_{\mu-1}(x)$. For each $j = 1, \dots, \mu - 2$ we take the scalar product of the equation (32) and the vector-function $\mathbf{v}_{\mu-1-j}(x)$. Summation of the results yields the expression

$$\sum_{j=0}^{\mu-1}\sum_{r=0}^{j}\frac{1}{r!}\langle\partial_\lambda^r\mathbf{L}\mathbf{u}_{j-r}, \mathbf{v}_{\mu-1-j}\rangle = 0. \tag{33}$$

Applying the Lagrange identity (6), (14) and taking into account relation (16), we transform (33) to the form

$$\sum_{j=0}^{\mu-1}\langle\mathbf{u}_{\mu-1-j}, \sum_{r=0}^{j}\frac{1}{r!}\partial_\lambda^r\mathbf{L}^\dagger\mathbf{v}_{j-r}\rangle + \sum_{k=0}^{\mu-1}\sum_{j=0}^{\mu-1-k}\left[\sum_{r=0}^{j}\left(\frac{1}{r!}\partial_\lambda^r\mathfrak{V}\mathfrak{v}_{j-r}\right)^*\right]\frac{\partial_\lambda^k\widetilde{\mathfrak{U}}}{k!}\mathfrak{u}_{\mu-1-j-k} = 0. \tag{34}$$

Equation (34) is satisfied in case when the vector-functions $\mathbf{v}_0(x), \mathbf{v}_1(x), \dots, \mathbf{v}_{\mu-1}(x)$ originate the Keldysh chain of the adjoint boundary value problem, corresponding to the $\mu$-fold eigenvalue $\bar{\lambda}_0$ [16,17]

$$\mathbf{L}_0^\dagger\mathbf{v}_0 = 0, \quad \mathfrak{V}_0\mathfrak{v}_0 = 0, \tag{35}$$

$$\mathbf{L}_0^\dagger\mathbf{v}_j = -\sum_{r=1}^{j}\frac{1}{r!}\partial_\lambda^r\mathbf{L}^\dagger\mathbf{v}_{j-r}, \quad \mathfrak{V}_0\mathfrak{v}_j = -\sum_{r=1}^{j}\frac{1}{r!}\partial_\lambda^r\mathfrak{V}\mathfrak{v}_{j-r}. \tag{36}$$

Taking the scalar product of Eq. (32) and the vector $\mathbf{v}_0$ and employing the expressions (6), (14) we arrive at the orthogonality conditions

$$\sum_{r=1}^{j}\frac{1}{r!}\left[\langle\partial_\lambda^r\mathbf{L}\mathbf{u}_{j-r}, \mathbf{v}_0\rangle + \ \mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\partial_\lambda^r\mathfrak{U}\mathfrak{u}_{j-r}\right] = 0, \quad j = 1, \dots, \mu - 1. \tag{37}$$

Substituting into Eqs. (1) the Newton–Puiseux series for the perturbed eigenvalue $\lambda(\varepsilon)$ and eigenvector $\mathbf{u}(\varepsilon)$ [16,17,31,32]

$$\lambda = \lambda_0 + \lambda_1\varepsilon^{1/\mu} + \cdots, \quad \mathbf{u} = \mathbf{w}_0 + \mathbf{w}_1\varepsilon^{1/\mu} + \cdots, \tag{38}$$

where $\mathbf{w}_0 = \mathbf{u}_0$, taking into account expansions (17) and (38) and collecting terms with the same powers of $\varepsilon$, yields $\mu - 1$ boundary value problems serving for determining the functions $\mathbf{w}_r$, $r = 1, 2, \dots, \mu - 1$

$$\mathbf{L}_0\mathbf{w}_r = -\sum_{j=0}^{r-1}\left(\sum_{\sigma=1}^{r-j}\frac{1}{\sigma!}\mathbf{L}_{\sigma 0}\sum_{|\alpha|_\sigma=r-j}\lambda_{\alpha_1}\dots\lambda_{\alpha_\sigma}\right)\mathbf{w}_j, \tag{39}$$

$$\mathfrak{U}_0\mathfrak{w}_r = -\sum_{j=0}^{r-1}\sum_{\sigma=1}^{r-j}\left(\sum_{|\alpha|_\sigma=r-j}\lambda_{\alpha_1}\dots\lambda_{\alpha_\sigma}\right)\frac{1}{\sigma!}\mathfrak{U}_{\sigma 0}\mathfrak{w}_j, \tag{40}$$

where $|\alpha|_\sigma = \alpha_1 + \cdots + \alpha_\sigma$ and $\alpha_1, \dots, \alpha_{\mu-1}$ are positive integers. For the existence and convergence of expansions (38) one can consult the work [28].

The vector-function $\mathbf{w}_\mu(x)$ is a solution of the boundary value problem

$$\mathbf{L}_0\mathbf{w}_\mu = -\mathbf{L}_{01}\mathbf{w}_0 - \sum_{j=0}^{\mu-1}\left(\sum_{\sigma=1}^{\mu-j}\frac{1}{\sigma!}\mathbf{L}_{\sigma 0}\sum_{|\alpha|_\sigma=\mu-j}\lambda_{\alpha_1}\dots\lambda_{\alpha_\sigma}\right)\mathbf{w}_j, \tag{41}$$

$$\mathfrak{U}_0\mathfrak{w}_\mu = -\mathfrak{U}_{01}\mathfrak{w}_0 - \sum_{j=0}^{\mu-1}\sum_{\sigma=1}^{\mu-j}\left(\sum_{|\alpha|_\sigma=\mu-j}\lambda_{\alpha_1}\dots\lambda_{\alpha_\sigma}\right)\frac{1}{\sigma!}\mathfrak{U}_{\sigma 0}\mathfrak{w}_j. \tag{42}$$

Comparing Eqs. (41) and (42) with the expressions (32) we find the first $\mu-1$ functions $\mathbf{w}_r$ in the expansions (38)

$$\mathbf{w}_r = \sum_{j=1}^{r}\mathbf{u}_j\sum_{|\alpha|_j=r}\lambda_{\alpha_1}\dots\lambda_{\alpha_j}. \tag{43}$$

With the vectors (43) we transform Eqs. (41) and (42) into

$$\mathbf{L}_0\mathbf{w}_\mu = -\mathbf{L}_{01}\mathbf{u}_0 - \lambda_1^\mu\sum_{r=1}^{\mu}\frac{1}{r!}\partial_\lambda^r\mathbf{L}\mathbf{u}_{\mu-r} + \sum_{j=1}^{\mu-1}\mathbf{L}_0\mathbf{u}_j\sum_{|\alpha|_j=\mu}\lambda_{\alpha_1}\dots\lambda_{\alpha_j}, \tag{44}$$

$$\mathfrak{U}_0\mathfrak{w}_\mu = -\mathfrak{U}_1\mathfrak{u}_0 - \lambda_1^\mu\sum_{r=1}^{\mu}\frac{1}{r!}\partial_\lambda^r\mathfrak{U}\mathfrak{u}_{\mu-r} + \sum_{j=1}^{\mu-1}\mathfrak{U}_0\mathfrak{u}_j\sum_{|\alpha|_j=\mu}\lambda_{\alpha_1}\dots\lambda_{\alpha_j}. \tag{45}$$

Applying the expression following from the Lagrange formula

$$\begin{aligned}\langle\mathbf{L}_0\mathbf{w}_\mu,\mathbf{v}_0\rangle &= \mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_{01}\mathfrak{u}_0 + \lambda_1^\mu\sum_{r=1}^{\mu}\frac{1}{r!}\mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_{r0}\mathfrak{u}_{\mu-r}\\ &\quad-\sum_{j=1}^{\mu-1}\mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_0\mathfrak{u}_j\sum_{|\alpha|_j=\mu}\lambda_{\alpha_1}\dots\lambda_{\alpha_j},\end{aligned} \tag{46}$$

and taking into account the equations for the adjoint Keldysh chain (35) and (36) yields the coefficient $\lambda_1$ in (38). Hence, the splitting of the $\mu$-fold non-derogatory eigenvalue $\lambda_0$ due to perturbation of the parameters $\mathbf{p}=\mathbf{p}(\varepsilon)$ is described by the following expression, generalizing the results of the works [16,17,42,43]

$$\lambda = \lambda_0 + \sqrt[\mu]{-\varepsilon\frac{\langle\mathbf{L}_{01}\mathbf{u}_0,\mathbf{v}_0\rangle + \mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_{01}\mathfrak{u}_0}{\sum_{r=1}^{\mu}\frac{1}{r!}(\langle\mathbf{L}_{r0}\mathbf{u}_{\mu-r},\mathbf{v}_0\rangle + \mathfrak{v}_0^*\widetilde{\mathfrak{V}}_0^*\mathfrak{U}_{r0}\mathfrak{u}_{\mu-r})}} + o(\varepsilon^{\frac{1}{\mu}}). \tag{47}$$

For $\mu=1$, Eq. (47) is reduced to Eq. (30) for a simple eigenvalue.

## 4. Example 1: a rotating circular string

Consider a circular string of displacement $W(\psi,\tau)$, radius $r$, and mass per unit length $\rho$ that rotates with the speed $\gamma$ and passes at $\psi=0$ through a massless eyelet supported by the spring with the stiffness $K$, as shown in Fig. 1a. Introducing the non-dimensional variables and parameters

$$t = \frac{\tau}{r}\sqrt{\frac{P}{\rho}},\quad w=\frac{W}{r},\quad \Omega=\gamma r\sqrt{\frac{\rho}{P}},\quad k=\frac{Kr}{P},\quad \varphi=\frac{\psi}{2\pi}, \tag{48}$$

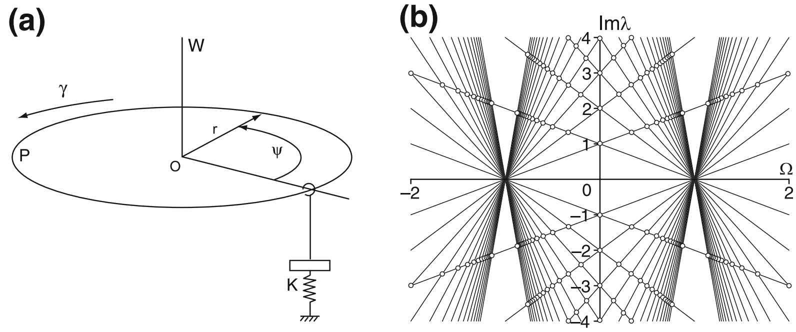


Fig. 1. A rotating circular string and 30 modes of its spectral mesh

and assuming $w(\varphi, t) = u(\varphi)\exp(\lambda t)$ we arrive at the non-self-adjoint boundary eigenvalue problem for a matrix $(N = 2, m = 1)$ differential operator [9]

$$\mathbf{Lu} := \mathbf{l}_0 \partial_\varphi \mathbf{u} + \mathbf{l}_1 \mathbf{u} = 0, \quad \mathfrak{U}\mathfrak{u} := [\mathfrak{A}, \mathfrak{B}]\mathfrak{u} = 0, \tag{49}$$

where

$$\mathbf{l}_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1-\Omega^2 \end{pmatrix}, \quad \mathbf{l}_1 = -\begin{pmatrix} 0 & 1 \\ 4\pi^2\lambda^2 & 4\pi\Omega\lambda \end{pmatrix}, \quad \mathfrak{A} = \begin{pmatrix} 1 & 0 \\ \frac{2\pi k}{\Omega^2-1} & 1 \end{pmatrix}, \quad \mathfrak{B} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{50}$$

The parameters $\Omega$ and $k$ express the speed of rotation and the stiffness coefficient.

With $\mathbf{u} = C_1(1, \frac{2\pi\lambda}{1-\Omega})^T \exp(\frac{2\pi\varphi\lambda}{1-\Omega}) + C_2(1, -\frac{2\pi\lambda}{1+\Omega})^T \exp(-\frac{2\pi\varphi\lambda}{1+\Omega})$ assumed as a solution of equation $\mathbf{Lu} = 0$, the characteristic equation follows from the boundary conditions $[\mathfrak{A}, \mathfrak{B}]\mathfrak{u} = 0$

$$k \sinh \frac{2\pi\lambda}{1-\Omega^2} - 4\lambda \sin \frac{\pi\lambda}{i(1-\Omega)} \sin \frac{\pi\lambda}{i(1+\Omega)} = 0, \tag{51}$$

and the relation between the coefficients $C_1$ and $C_2$ of the eigenvector $\mathbf{u}$

$$\left(1 - e^{-\frac{2\lambda\pi}{\Omega-1}}\right) C_1 + \left(1 - e^{-\frac{2\lambda\pi}{\Omega+1}}\right) C_2 = 0. \tag{52}$$

For the unconstrained rotating string with $k = 0$ the eigenvectors $\mathbf{v}$ and $\mathbf{u}$ of the adjoint problems, corresponding to purely imaginary eigenvalue $\lambda$ and $\bar{\lambda}$, coincide. The eigenvalues $\lambda_n^{\pm} = in(1 \pm \Omega)$, $n \in \mathbb{Z}$, form the spectral mesh in the plane $(\Omega, \mathrm{Im}\lambda)$, Fig. 1b. The lines $\lambda_n^{\varepsilon} = in(1 + \varepsilon\Omega)$ and $\lambda_m^{\delta} = im(1 + \delta\Omega)$, where $\varepsilon, \delta = \pm 1$, intersect each other at the node $(\Omega_{mn}^{\varepsilon\delta}, \lambda_{mn}^{\varepsilon\delta})$ with

$$\Omega_{mn}^{\varepsilon\delta} = \frac{n-m}{m\delta - n\varepsilon}, \quad \lambda_{mn}^{\varepsilon\delta} = \frac{inm(\delta - \varepsilon)}{m\delta - n\varepsilon}, \tag{53}$$

where the double eigenvalue $\lambda_{mn}^{\varepsilon\delta}$ has two orthogonal eigenvectors

$$\mathbf{u}_n^{\varepsilon} = \begin{pmatrix} 1 \\ -i\varepsilon 2\pi n \end{pmatrix} e^{-i\varepsilon 2\pi n\varphi}, \quad \mathbf{u}_m^{\delta} = \begin{pmatrix} 1 \\ -i\delta 2\pi m \end{pmatrix} e^{-i\delta 2\pi m\varphi}. \tag{54}$$

Using the perturbation formulas (28) and (29) for semi-simple eigenvalues with the eigenelements (53) and (54) we find an asymptotic expression for the eigenvalues originated after the splitting of the double eigenvalues $\lambda_{nm}^{\varepsilon\delta}$ at the nodes of the spectral mesh in the subcritical region $|\Omega| < 1$ ($\varepsilon < 0$, $\delta > 0$ and $m > n > 0$) due to interaction of the rotating string with the external spring

$$\lambda = \lambda_{nm}^{\varepsilon\delta} + i\frac{m-n}{2}\Delta\Omega + i\frac{n+m}{8\pi nm}k \pm i\sqrt{\frac{k^2}{16\pi^2 nm} + \left(\frac{m-n}{8\pi mn}k - \frac{m+n}{2}\Delta\Omega\right)^2}. \tag{55}$$

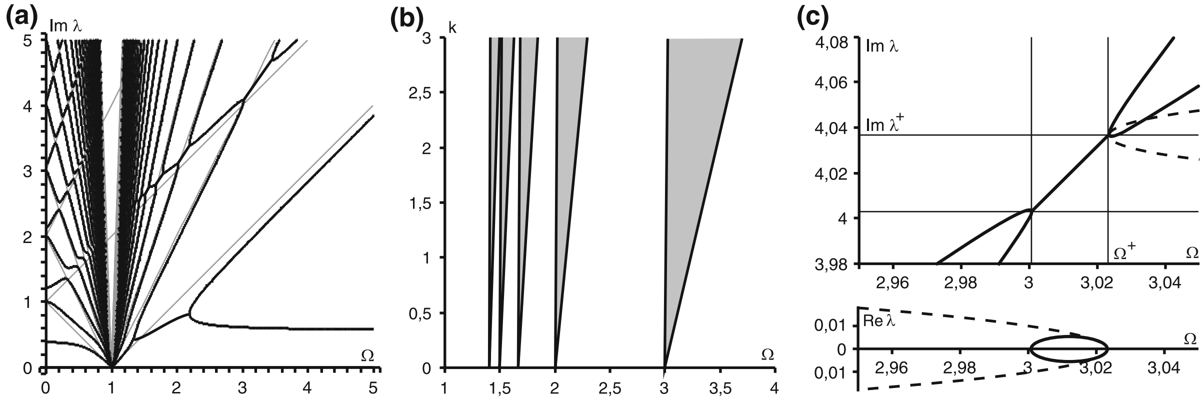


FIG. 2. **a** Deformation of the spectral mesh of the rotating string interacting with the external spring with $k = 2$; **b** approximation to the corresponding tongues of oscillatory instability; **c** comparing numerically calculated eigenvalue curves (*full lines*) with the approximation (60) (*dashed lines*) for $k = 0.1$

In the supercritical region $|\Omega| > 1$ ($\varepsilon < 0$, $\delta > 0$ and $m > 0$, $n < 0$) we have

$$\lambda = \lambda_{nm}^{\varepsilon\delta} + i\frac{m+|n|}{2}\Delta\Omega + i\frac{|n|-m}{8\pi|n|m}k \pm \sqrt{\frac{k^2}{16\pi^2|n|m} - \left(\frac{|n|-m}{2}\Delta\Omega - \frac{m+|n|}{8\pi m|n|}k\right)^2}. \tag{56}$$

Therefore, for $|\Omega| < 1$ the spectral mesh collapses into separated curves demonstrating avoided crossings; for $|\Omega| > 1$ the eigenvalue branches overlap forming the bubbles of instability with eigenvalues having positive real parts, see Fig. 2a. From (56) a linear approximation follows to the boundary of the domains of supercritical flutter instability in the plane $(\Omega, k)$ (gray resonance tongues in Fig. 2b)

$$k = \frac{4\pi|n|m(|n|-m)}{(\sqrt{|n|} \pm \sqrt{m})^2}\left(\Omega - \frac{|n|+m}{|n|-m}\right). \tag{57}$$

The stability boundary consists of exceptional points at which there exist double purely imaginary eigenvalues with the Keldysh chain. Their approximate locations in the $(\Omega, \mathrm{Im}\lambda)$-plane follow from the expressions (56) and (57). For the resonance tongue, originated at the diabolical point $(\Omega = 3, \mathrm{Im}\lambda = 4)$, the approximation to the loci of the exceptional points are

$$\Omega^{\pm} = 3 + \frac{3 \pm 2\sqrt{2}}{8\pi}k, \quad \lambda^{\pm} = i\left(4 + \frac{5 \pm 3\sqrt{2}}{8\pi}k\right). \tag{58}$$

For small values of $k$ the coordinates (58) are very close to that found from the numerical solution of the characteristic equation (51), as is illustrated by Fig. (2)c.

The double eigenvalue $\lambda^+$ at $\Omega = \Omega^+$ has an eigenvector $\mathbf{u}_0 = (u_{01}, u_{02})^T$ and associated vector $\mathbf{u}_1 = (u_{11}, u_{12})^T$. For a given $k$ the eigenvalue $\lambda^+$ splits with the variation of $\Omega$ in accordance to the formula (47), which now reads as

$$\lambda = \lambda^+ \pm \sqrt{\frac{(1-\Omega^{+2})\int_0^1 \bar{v}_{02}(\Omega^+ u_{02}' + 2\pi\lambda^+ u_{02})d\varphi + 2\pi k\Omega^+ \bar{v}_{02}(0)u_{01}(0)}{-2\pi(1-\Omega^{+2})\int_0^1 \bar{v}_{02}(2\pi\lambda^+ u_{11} + \Omega^+ u_{12} + \pi u_{01})d\varphi}(\Omega - \Omega^+)}. \tag{59}$$

With the vectors $\mathbf{u}_0$ and $\mathbf{u}_1$, and with the left eigenvector $\mathbf{v}_0 = (v_{01}, v_{02})^T$ calculated for $k = 0.1$ at the exceptional point $(\Omega^+, \mathrm{Im}\lambda^+)$, the formula (59) yields

$$\lambda = \lambda^+ \pm \sqrt{-0.004349(\Omega - \Omega^+)}. \tag{60}$$

In Fig. 2c the dashed lines correspond to approximation (60) and the full lines show the numerical solution of equation (51) which on this scale is undistinguishable from the approximation (56). The deformation patterns of the spectral mesh and first-order approximations of the instability tongues obtained by the perturbation technique are in a good qualitative and quantitative agreement with the results of numerical calculations of [9].

## 5. Example 2: MHD $\alpha^2$-dynamo

Consider a non-self-adjoint boundary eigenvalue problem ($N=2, m=2$) appearing in the theory of MHD $\alpha^2$-dynamo (see [3,7,47,48] and references therein for the history and state of the art of the problem)

$$\mathbf{L}\mathbf{u} := \mathbf{l}_0\partial_x^2\mathbf{u} + \mathbf{l}_1\partial_x\mathbf{u} + \mathbf{l}_2\mathbf{u} = 0, \quad \mathfrak{U}\mathfrak{u} := [\mathfrak{A}, \mathfrak{B}]\mathfrak{u} = 0, \tag{61}$$

with the matrices of the differential expression

$$\mathbf{l}_0 = \begin{pmatrix} 1 & 0 \\ -\alpha(x) & 1 \end{pmatrix}, \quad \mathbf{l}_1 = \partial_x\mathbf{l}_0, \quad \mathbf{l}_2 = \begin{pmatrix} -\frac{l(l+1)}{x^2} - \lambda & \alpha(x) \\ \alpha(x)\frac{l(l+1)}{x^2} & -\frac{l(l+1)}{x^2} - \lambda \end{pmatrix}, \tag{62}$$

and of the boundary conditions

$$\mathfrak{A} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathfrak{B} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \beta l + 1 - \beta & 0 & \beta & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \tag{63}$$

where it is assumed that $\alpha(x) = \alpha_0 + \gamma\Delta\alpha(x)$ with $\int_0^1 \Delta\alpha(x)\mathrm{d}x = 0$. For fixed $\Delta\alpha(x)$ the differential expression depends on the parameters $\alpha_0$ and $\gamma$, while $\beta$ interpolates between the idealistic ($\beta = 0$) and physically realistic ($\beta = 1$) boundary conditions [7,49].

The matrix $\mathfrak{V}$ of the boundary conditions and auxiliary matrix $\widetilde{\mathfrak{V}}$ for the adjoint differential expression $\mathbf{L}^\dagger\mathbf{v} = \mathbf{l}_0^*\partial_x^2\mathbf{v} - \mathbf{l}_1^*\partial_x\mathbf{v} + \mathbf{l}_2^*\mathbf{v}$ follow from the formula (15) where the $4\times4$ matrices $\widetilde{\mathfrak{A}}$, and $\widetilde{\mathfrak{B}}$ are chosen as

$$\widetilde{\mathfrak{A}} = \begin{pmatrix} 0 & \mathbf{I} \\ 0 & 0 \end{pmatrix}, \quad \widetilde{\mathfrak{B}} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{I} \end{pmatrix}. \tag{64}$$

In the following we assume that $l = 0$ and interpret $\beta$ and $\gamma$ as perturbing parameters. It is known [7] that for $\beta = 0$ and $\gamma = 0$ the spectrum of the unperturbed eigenvalue problem (61) forms the spectral mesh in the plane $(\alpha_0, \lambda)$ shown by the dashed lines in Fig. 3a. The eigenelements of the spectral mesh are

$$\lambda_n^\varepsilon = -(\pi n)^2 + \varepsilon\alpha_0\pi n, \quad \lambda_m^\delta = -(\pi m)^2 + \delta\alpha_0\pi m, \quad \varepsilon, \delta = \pm1, \tag{65}$$

$$\mathbf{u}_n^\varepsilon = \begin{pmatrix} 1 \\ \varepsilon\pi n \end{pmatrix}\sin(n\pi r), \quad \mathbf{u}_m^\delta = \begin{pmatrix} 1 \\ \delta\pi m \end{pmatrix}\sin(m\pi r). \tag{66}$$

The branches (65) intersect and originate a double semi-simple eigenvalue with two linearly independent eigenvectors (66) at the node $(\alpha_0^\nu, \lambda_0^\nu)$, where [7]

$$\lambda_0^\nu = \varepsilon\delta\pi^2 nm, \quad \alpha_0^\nu = \varepsilon\pi n + \delta\pi m. \tag{67}$$

Taking into account that the components of the eigenfunctions of the adjoint problems are related as $\bar{v}_2 = u_1$ and $\bar{v}_1 = u_2$, we find from Eqs. (28) and (29) the asymptotic formula for the perturbed eigenvalues, originating after the splitting of the double semi-simple eigenvalues at the nodes of the spectral mesh

$$\lambda = \lambda_0^\nu - \varepsilon\delta\pi^2 mn\beta + \frac{\pi}{2}(\delta m + \varepsilon n)\Delta\alpha_0 \pm \frac{\pi}{2}\sqrt{D}, \tag{68}$$

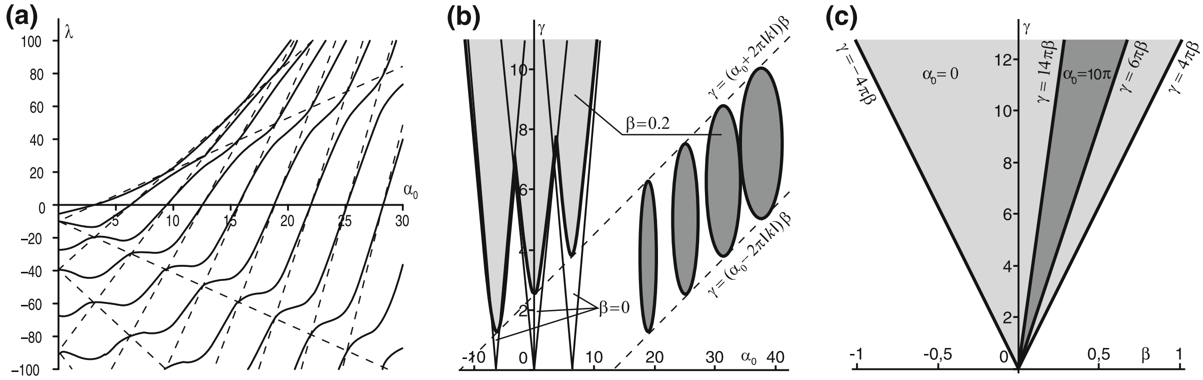


FIG. 3. $l = 0$: **a** deformation of the spectral mesh for $\gamma = 0$ and $\beta \in (0, 1)$; **b**, **c** approximation of the resonance tongues for $\Delta\alpha(x) = \cos(4\pi x)$ and (*white, light gray*) $\lambda_0^\nu < 0$ or (*dark gray*) $\lambda_0^\nu > 0$

where $\Delta\alpha_0 := \alpha_0 - \alpha_0^\nu$, $\Delta\alpha := \int_0^1 \Delta\alpha(x)\cos((\varepsilon n - \delta m)\pi x)\mathrm{d}x$, and

$$D := ((\varepsilon n - \delta m)\Delta\alpha_0)^2 + mn\left((\varepsilon+\delta)\gamma\Delta\alpha - (-1)^{n+m}(n+m)\beta\pi\right)^2 \\ -mn\left((\varepsilon-\delta)\gamma\Delta\alpha - (-1)^{n-m}(n-m)\beta\pi\right)^2. \tag{69}$$

When $\gamma = 0$ and $\Delta\alpha_0 = 0$, one of the two simple eigenvalues (68) remains unshifted in the first order of the perturbation theory with respect to $\beta$: $\lambda = \lambda_0^\nu$. The sign of the first-order increment to another eigenvalue $\lambda = \lambda_0^\nu - 2\lambda_0^\nu\beta$ depends on the sign of $\lambda_0^\nu$, which is directly determined by the Krein signature of the modes involved in the crossing [7]. This is in full agreement with the numerically computed roots of the characteristic equation derived in [49] for the problem (61) with $\alpha(x) = \alpha_0 = const$ and $l = 0$

$$(1-\beta)\eta\left[\cos(\eta) - \cos(\alpha_0)\right] + 2\beta\lambda\sin(\eta) = 0, \tag{70}$$

where $\eta(\alpha_0, \lambda) = \sqrt{\alpha_0^2 - 4\lambda}$, shown in Fig. 3a. Therefore, under variation of the parameter $\beta$ in the boundary conditions the eigenvalues remain real. An additional parameter $\gamma$ is required to create complex eigenvalues. This happens when $D < 0$ in (68).

The inequality $D < 0$ defines the inner part of the cone $D = 0$ in the $(\alpha_0, \beta, \gamma)$-space. The part of the cone corresponding to $\mathrm{Re}\lambda > 0$ (oscillatory dynamo) is selected by the condition $2\lambda_0^\nu - \varepsilon\delta 2\pi^2 mn\beta + \pi(\delta m + \varepsilon n)\Delta\alpha_0 > 0$. The conical zones develop according to the resonance selection rules discovered in [7]. For example, if $\Delta\alpha(x) = \cos(2\pi k x)$, $k \in \mathbb{Z}$, then

$$\Delta\alpha = \int_0^1 \cos(2\pi k x)\cos((\varepsilon n - \delta m)\pi x)\mathrm{d}x = \begin{cases} 1/2, & 2k = \pm(\varepsilon n - \delta m) \\ 0, & 2k \neq \pm(\varepsilon n - \delta m) \end{cases} \tag{71}$$

There exist a set of $2|k| - 1$ cones for $\lambda_0^\nu < 0$ ($\varepsilon\delta < 0$) and a set with countably infinite number of cones for $\lambda_0^\nu > 0$ ($\varepsilon\delta > 0$). Only the cones of the first set intersect with the plane $\beta = 0$. The cross-sections of the domains of oscillatory dynamo are situated symmetrically with respect to the $\gamma$-axis

$$(\alpha_0 \pm 2\pi(n - |k|))^2 < \frac{\gamma^2}{4}\left[1 - \left(\frac{n - |k|}{|k|}\right)^2\right], \quad n = 1, 2, \ldots, |k|. \tag{72}$$

For $k = 2$ there are three resonant tongues: $4\alpha_0^2 < \gamma^2$ and $16\left(\alpha_0 \pm 2\pi\right)^2 < 3\gamma^2$, which are shown white in Fig. 3b.

When $\beta \neq 0$ the tongues (72) in the plane $(\alpha_0, \gamma)$, corresponding to $\lambda_0^\nu < 0$, deform into the cross-sections of the cones bounded by hyperbolic curves (black thick lines in Fig. 3b)

$$-4k^2(\alpha_0 \pm 2\pi(n-|k|))^2 + n(2|k|-n)(\gamma \pm 2\pi(n-|k|)\beta)^2 > n(2|k|-n)4\pi^2\beta^2k^2, \quad (73)$$

with $n = 1, 2, \ldots, |k|$. Since $n \leq |k|$, the lines $\gamma = \pm 2\pi n\beta$ and $\gamma = \pm 2\pi(n - 2|k|)\beta$, bounding the cross-sections of the 3D cones by the plane $\alpha_0 = \alpha_0^{(\nu)} = \pm 2\pi(n - |k|)$, always have the slopes of different sign, Fig. 3c. This allows decaying oscillatory modes for $\beta = 0$ due to variation of $\gamma$ only. For $k = 2$ the approximation (73) is shown light gray in Fig. 3b and c.

In the $\beta \neq 0$-plane cross-sections of the cones, corresponding to $\lambda_0^\nu > 0$, have the form of the ellipses, shown dark gray in Fig. 3b

$$4k^2\left(\alpha_0 \pm 2\pi(n+|k|)\right)^2 + n(2|k|+n)\left(\gamma \pm 2\pi(n+|k|)\beta\right)^2 < n(2|k|+n)4\pi^2\beta^2k^2, \quad (74)$$

where $n = 1, 2, \ldots$. Inside the ellipses there exist eigenvalues with positive real parts exciting the *oscillatory dynamo* regime. In the $(\beta \neq 0)-$plane the ellipses (74) are located inside the strip with boundaries $\gamma = (\alpha_0 \pm 2\pi|k|)\beta$ (dashed lines in Fig. 3b), while the hyperbolas (73) lie outside this strip. Hence, the amplitude $\gamma$ of the resonant perturbation of the $\alpha$-profile $\gamma\Delta\alpha(x)$ is limited both from below and from above in agreement with the numerical findings of [47]. Moreover, since in the plane $\alpha_0 = \pm 2\pi(n + |k|)$ the boundary lines $\gamma = \pm 2\pi n\beta$ and $\gamma = \pm 2\pi(n + 2|k|)\beta$ have slopes of the same sign, the $\gamma$-axis does not belong to the instability domains, showing that for growing oscillatory modes the parameters $\beta$ and $\gamma$ have to be taken in a prescribed proportion, see Fig. 3c. We note that the analytical results confirm the numerical simulations of the earlier works [3,47].

## 6. Conclusion

A perturbative approach to multiparameter non-self-adjoint boundary eigenvalue problems for operator matrices is developed in the form convenient for implementation in the computer algebra systems for an automatic calculation of the adjoint boundary conditions and coefficients in the perturbation series for simple and multiple eigenvalues and their eigenvectors. The approach is aimed at applications requiring frequent switches from one set of boundary conditions to another. Two studies of the onset of instability in rotating continua under symmetry-breaking perturbations demonstrate the efficiency of the proposed technique.

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O. N. Kirillov
Dynamics and Vibrations Group
Department of Mechanical Engineering
Technische Universität Darmstadt
Hochschulstr. 1
64289 Darmstadt
Germany
e-mail: kirillov@dyn.tu-darmstadt.de

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