WKB instability thresholds of the magnetized cylindrical Couette-Taylor flow in helical magnetic fields

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Summary

We consider a cylindrical Couette-Taylor (CT) flow of an electrically conducting viscous and resistive fluid in an external helical magnetic field. Local stability of the flow is studied with respect to three-dimensional perturbations within a short-wavelength approximation. Maximization of the critical Rossby number at the instability threshold is performed with respect to the non-dimensional parameters of the problem characterizing hydrodynamic and magnetic effects. Quite surprisingly, it is found that the critical Rossby number at the threshold of magnetorotational instability in the case of infinitesimally small magnetic Prandtl number is universally bounded from above by a quantity $2-2\sqrt{2}$ known as the Liu limit, which is below that of Keplerian rotation $(-3/4)$.

We consider the rotational flow of an incompressible electrically conducting viscous (ν) and resistive (η) fluid in the gap between the radii R_1 and $R_2 > R_1$, with an imposed magnetic field sustained by currents external to the fluid [1, 2, 3]. Introducing cylindrical coordinates (R, ϕ, z) we investigate stability of a steady-state background liquid flow with the angular velocity profile $\Omega(R)$ in a helical background magnetic field (a magnetized CT-flow)

$$\mathbf{u}_0 = R\,\Omega(R)\,\mathbf{e}_\phi, \quad p = p_0(R), \quad \mathbf{B}_0 = B_\phi^0(R)\mathbf{e}_\phi + B_z^0\mathbf{e}_z. \tag{1}$$

Linearizing nonlinear Navier-Stokes equations coupled with the induction equation in the vicinity of the stationary solution (1) by assuming general perturbations $\mathbf{u} = \mathbf{u}_0 + \mathbf{u}'$, $p = p_0 + p'$, and $\mathbf{B} = \mathbf{B}_0 + \mathbf{B}'$, we find

$$\begin{aligned}(\partial_t + \mathcal{U} + \mathbf{u}_0\cdot\nabla)\mathbf{u}' + \frac{1}{\rho}\nabla p' + \frac{1}{\rho\mu_0}\mathbf{B}_0\times(\nabla\times\mathbf{B}') &= \nu\nabla^2\mathbf{u}',\\ (\partial_t - \mathcal{U} + \mathbf{u}_0\cdot\nabla)\mathbf{B}' + (\mathcal{B} - \mathbf{B}_0\cdot\nabla)\mathbf{u}' &= \eta\nabla^2\mathbf{B}',\end{aligned} \tag{2}$$

where ρ is the density of the fluid and μ_0 is the magnetic permeability of free space.

We seek for solutions of the linearized equations (2) in the form of a rapid oscillation with slowly varying amplitude of its envelope [4, 5]

$$\begin{aligned}\mathbf{u}'(\mathbf{x},t,\epsilon) &= e^{i\Phi(\mathbf{x},t)/\epsilon}\left(\mathbf{u}^{(0)}(\mathbf{x},t) + \epsilon\mathbf{u}^{(1)}(\mathbf{x},t)\right) + \epsilon\mathbf{u}^{(r)}(\mathbf{x},t),\\ \mathbf{B}'(\mathbf{x},t,\epsilon) &= e^{i\Phi(\mathbf{x},t)/\epsilon}\left(\mathbf{B}^{(0)}(\mathbf{x},t) + \epsilon\mathbf{B}^{(1)}(\mathbf{x},t)\right) + \epsilon\mathbf{B}^{(r)}(\mathbf{x},t),\\ p'(\mathbf{x},t,\epsilon) &= e^{i\Phi(\mathbf{x},t)/\epsilon}\left(p^{(0)}(\mathbf{x},t) + \epsilon p^{(1)}(\mathbf{x},t)\right) + \epsilon p^{(r)}(\mathbf{x},t),\end{aligned} \tag{3}$$

with $i=\sqrt{-1}$, $0<\epsilon\ll 1$ a small parameter, $\mathbf{x}$ denoting the vector of coordinates, Φ a real-valued phase of the oscillations, and $\mathbf{u}^{(j)}$, $p^{(j)}$, and $\mathbf{B}^{(j)}$, where $j=0,1,r$, the complex-valued amplitudes corresponding to fluid velocity, pressure, and magnetic field. Assuming $\nu=\epsilon^2\tilde{\nu}$ and $\eta=\epsilon^2\tilde{\eta}$ [6] and introducing the derivative

$$\frac{D}{Dt} := \partial_t + \mathbf{u}_0\cdot\nabla \tag{4}$$

along the fluid stream lines, we substitute expressions (3) into equations (2), collect terms at ϵ^{-1} and ϵ^0 and expand the cross products, which yields the equation for the time evolution of the wave vector $\mathbf{k} := \nabla\Phi$

$$\frac{D\mathbf{k}}{Dt} = -\mathcal{U}^T\mathbf{k}, \tag{5}$$

and, given the wave vector $\mathbf{k}(t)$, the transport equations, determining the evolution of the amplitudes $\mathbf{u}^{(0)}$ and $\mathbf{B}^{(0)}$

$$\begin{aligned}\frac{D\mathbf{u}^{(0)}}{Dt} &= \left[2\frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}-\mathcal{I}\right]\mathcal{U}\mathbf{u}^{(0)}-\tilde{\nu}|\mathbf{k}|^2\mathbf{u}^{(0)}+\frac{1}{\rho\mu_0}\left[\mathcal{I}-\frac{\mathbf{k}\mathbf{k}^T}{|\mathbf{k}|^2}\right](\mathcal{B}+\mathbf{B}_0\cdot\nabla)\mathbf{B}^{(0)}\\ \frac{D\mathbf{B}^{(0)}}{Dt} &= \mathcal{U}\mathbf{B}^{(0)}-\tilde{\eta}|\mathbf{k}|^2\mathbf{B}^{(0)}-(\mathcal{B}-\mathbf{B}_0\cdot\nabla)\mathbf{u}^{(0)},\end{aligned}\tag{6}$$

where $\mathcal{U}(R) := \nabla\mathbf{u}_0$, $\mathcal{B}(R) := \nabla\mathbf{B}_0$, and $\mathcal{I}$ is the 3×3 identity matrix. In the absence of the magnetic field Eqs. (6) are reduced to the hydrodynamical ones derived in [5, 6].

Considering a solution to Eqs. (5) with $k_\phi \equiv 0$, so that the wavenumbers k_R and k_z are arbitrary constants [5] and denoting $|\mathbf{k}|^2 = k_R^2+k_z^2$ and $\alpha = k_z|\mathbf{k}|^{-1}$ we seek for the solutions of Eqs. (6) that are proportional to $e^{\gamma t+im\phi+ik_z z}$. This yields a dispersion relation $\det(\mathbf{H}(\lambda))=0$, with

$$\mathbf{H}=\begin{pmatrix} -\frac{im\mathrm{Re}\sqrt{\mathrm{Pm}}+\alpha\sqrt{\mathrm{Pm}}+\lambda\alpha}{\alpha\sqrt{\mathrm{Pm}}} & 2\alpha\mathrm{Re} & \frac{i\mathrm{Ha}(\alpha+m\beta)}{\alpha\sqrt{\mathrm{Pm}}} & -\frac{2\alpha\beta\mathrm{Ha}}{\sqrt{\mathrm{Pm}}}\\ -\frac{2\mathrm{Re}(1+\mathrm{Ro})}{\alpha} & -\frac{im\mathrm{Re}\sqrt{\mathrm{Pm}}+\alpha\sqrt{\mathrm{Pm}}+\lambda\alpha}{\alpha\sqrt{\mathrm{Pm}}} & 0 & \frac{i\mathrm{Ha}(\alpha+m\beta)}{\alpha\sqrt{\mathrm{Pm}}}\\ \frac{i\mathrm{Ha}(\alpha+m\beta)}{\alpha\sqrt{\mathrm{Pm}}} & 0 & -\frac{im\mathrm{Re}\mathrm{Pm}+\alpha+\lambda\alpha\sqrt{\mathrm{Pm}}}{\alpha\mathrm{Pm}} & 0\\ \frac{2\beta\mathrm{Ha}}{\alpha\sqrt{\mathrm{Pm}}} & \frac{i\mathrm{Ha}(\alpha+m\beta)}{\alpha\sqrt{\mathrm{Pm}}} & \frac{2\mathrm{Re}\mathrm{Ro}}{\alpha} & -\frac{im\mathrm{Re}\mathrm{Pm}+\alpha+\lambda\alpha\sqrt{\mathrm{Pm}}}{\alpha\mathrm{Pm}}\end{pmatrix},\tag{7}$$

where $\mathrm{Ro}=\frac{1}{2}\frac{R}{\Omega}\frac{d\Omega}{dR}$ is the Rossby number, $\mathrm{Pm}=\frac{\nu}{\eta}$ the magnetic Prandtl number, $\beta=\alpha\frac{\omega_{A_\phi}}{\omega_A}$ the ratio of the Alfvén frequencies, $\mathrm{Ha}=\alpha\frac{B_z^0}{k\sqrt{\mu_0\rho\nu\eta}}$ the Hartmann number, $\mathrm{Re}=\alpha\frac{\Omega}{\omega_\nu}$ the Reynolds number, $n=\frac{m}{\alpha}$, and $\gamma=\lambda\sqrt{\omega_\nu\omega_\eta}$, $\omega_\nu=\nu|\mathbf{k}|^2$, $\omega_\eta=\eta|\mathbf{k}|^2$, $\omega_A^2=\frac{k_z^2 {B_z^0}^2}{\rho\mu_0}$, and $\omega_{A_\phi}^2=\frac{(B_\phi^0)^2}{\rho\mu_0 R^2}$.

The onset of magnetorotational instability (MRI) corresponds to the transition of one of the roots λ of the dispersion relation to the right side of the complex plane. We prove that when $\mathrm{Pm}\to 0$, for all Ha and Re the Rossby number at the onset of the MRI does not exceed the value

$$\mathrm{Ro}(n,\beta)=\frac{4\beta^4+(\beta n+1)^4-(2\beta^2+(\beta n+1)^2)\sqrt{4\beta^4+(\beta n+1)^4}}{2\beta^2(\beta n+1)^2}\le 2-2\sqrt{2}.\tag{8}$$

From this universal behavior we are also led to the prediction that the instability will be governed by a mode with an azimuthal wavenumber that is proportional to the ratio of axial to azimuthal applied magnetic field, when this ratio becomes large and the Rossby number is close to the Liu limit. We note that in the astrophysical applications, the low Pm MRI versions could onset within segments of steepening shear in protoplanetary and accretion disks [7]. Besides, they were observed in the recent liquid metal experiments devoted to the study of MRI [8].

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