Optimization of Stability of a Flying Column

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1. Abstract
A non-uniform column moving in space under a tangential end force (as idealization of a flexible missile) is considered. It is supposed that the column can carry a nonstructural mass (a payload). This nonconservative (circulatory) system can lose stability by flutter or by divergence under certain conditions. Changing mass and/or stiffness distribution of the column or nonstructural mass distribution we can influence the critical end force. Two optimization problems with critical follower load as a design criterion are studied, mass distributions of the column and payload being taken as control functions.

The problem of optimal displacement of a concentrated mass along the uniform column is also considered. Since this problem possesses three parameters, stability diagrams in three- and two- parameter spaces are plotted. It is found that the optimal solution is attained at the singularity of a boundary of the stability domain.

Special attention is paid to singularities arising on the boundaries between stability and instability domains. For two-parametric linear circulatory systems of general type with finite degrees of freedom we formulate a theorem where full description of boundaries between stability, flutter and divergence domains is given; typical singularities of boundaries are listed, and explicit expressions for normals and tangent cones to the boundaries via left and right eigenvectors and associated vectors of the matrix of a circulatory system are presented.

2. Keywords
Optimization, Sensitivity analysis, Circulatory systems, Stability, Singularities of stability boundaries

3. Problem formulation
Consider a planar motion of a flexible non-uniform column under a tangential end force. It is assumed that the column carries a distributed payload of a given mass. The mass of the column is given and fixed. The separated dimensionless differential equation and the boundary conditions describing transverse vibrations of the column are

$$
\begin{gathered}
(\sigma(x)u'')'' + p\frac{M_{col}}{M}\left(u'\int_x^1 m(\xi)d\xi\right)' - m\omega^2 u = 0,\\
(\sigma(x)u'')\Big|_{x=0} = 0,\ \ (\sigma(x)u'')'\Big|_{x=0} = 0,\\
(\sigma(x)u'')\Big|_{x=1} = 0,\ \ (\sigma(x)u'')'\Big|_{x=1} = 0,
\end{gathered}
\tag{1}
$$

where x is a coordinate along the central line of the column, u is a deflection function, primes denote differentiation with respect to x, p is a follower force, ω is a natural frequency of the system under the follower force, $\sigma(x)$ is a bending stiffness, $m(x) = m_{col}(x) + m_{nm}(x)$, $m_{col}(x)$ is a mass of the column per unit length, $m_{nm}(x)$ is a nonstructural mass per unit length, $M_{col} = \int_0^1 m_{col}(x)dx = 1$ is a mass of the column, and $M = \int_0^1 m(x)dx$ is a total mass of the system.

First we consider a column without nonstructural mass, i.e. $m(x) = m_{col}(x)$. It is supposed that the column has circular cross-sections. This yields the relationship $\sigma(x) = m_{col}^2(x)$ between the bending stiffness and the mass distribution of the column.

Solving eigenvalue problem (1) for the given mass disribution $m(x) > 0$ under different values of parameter $p \ge 0$ we can find eigenvalues $\omega_j^2(p)$ as functions of the load parameter. The column is stable iff all ω_j^2 are positive and semi-simple eigenvalues. If one of the eigenvalues ω_j^2 changes the sign at $p = p_*^{stat}$ and then becomes negative, it means that the column loses stability statically (divergence). If two eigenvalues ω_j^2, ω_k^2, $j \ne k$ merge at $p = p_*^{dyn}$, i.e. $\omega_j^2 = \omega_k^2$, and then split to complex-conjugate eigenvalues, it means dynamical loss of stability (flutter). For the given $m(x)$ critical load p_* is defined as $\min\{p_*^{dyn}, p_*^{stat}\}$. Changing mass distribution we can influence the critical follower load.

We would like to find a mass distribution of the column which maximizes the critical follower force p_*, destabilizing the system either by flutter or by divergence. Hence, we formulate an optimization problem

$$p_*(m) \to \max_{m \in \Omega},$$
$$\Omega = \left\{ m(x): \int_0^1 m dx = 1, m(x) > 0, x \in [0,1] \right\}. \tag{2}$$

Consider now the case when a uniform column carries a distributed payload of a given mass. Then the mass and stiffness distributions of the column in (1) are constant: $m_{col}(x) \equiv 1$, $\sigma(x) \equiv 1$. Introducing a parameter $\kappa = \frac{M - M_{col}}{M}$, characterizing the part of the payload in the total mass of the system, we can express mass distribution of the system in the form

$$m(x) = 1 + \frac{\kappa}{1-\kappa}\mu(x), \quad \int_0^1 \mu dx = 1 .$$

Our aim is to find a nonstructural mass distribution $\mu(x)$ maximizing the critical end force $p_* = \min\left\{ p_*^{dyn}, p_*^{stat} \right\}$, assuming that admissible $\mu(x)$ is a bounded function. So, we formulate an optimization problem as

$$p_*(\mu) \to \max_{\mu \in \Omega},$$
$$\Omega = \left\{ \mu(x): \int_0^1 \mu dx = 1, 0 \le \mu(x) \le \mu_{up}, x \in [0,1] \right\}. \tag{3}$$

4. Method of solution and the results obtained

In both problems (2) and (3) optimal solutions are obtained numerically with the use of the gradient projection method. Eigenvalue problem adjoint to (1) is introduced to derive explicit expressions for gradient functions showing sensitivity of the critical flutter or divergence loads with respect to stiffness and mass rearrangements. To compute gradient functions it is necessary to know only solutions of main and adjoint eigenvalue problems at the critical point p_*.

Consider, for example, optimization problem (2). Due to variation of mass $m = m(x) + \delta m(x)$ eigenvalues, eigenfunctions, and critical load take increments $\delta\omega_*$, δu_*, δp_*. Substituting these increments into (1) yields after simple transformations the relationship $\delta p_* = \int_0^1 g(x)\delta m(x) dx$ between $\delta m(x)$ and δp_*. Function $g(x)$ is a gradient function of the critical load p_* with respect to variations of mass distribution of the column. Expressions for $g(x)$ are different in the cases $p_* = p_*^{dyn}$ and $p_* = p_*^{stat}$:

$$g_f(x) = \frac{\omega_*^2 u_* v_* - 2mu_*'' v_*'' + p_* v_*(0) u_*'(0) + p_* \int_0^x u_*' v_*' d\xi}{\int_0^1 v_* \left(u_*' \int_x^1 m(\xi) d\xi \right)' dx}, \tag{4}$$

$$g_d(x) = \frac{-2mu_*'' v_*'' + p_* v_*(0) u_*'(0) + p_* \int_0^x u_*' v_*' d\xi}{\int_0^1 v_* \left(u_*' \int_x^1 m(\xi) d\xi \right)' dx}. \tag{5}$$

Here $g_f(x)$ is the gradient function of flutter critical load, $g_d(x)$ is the gradient function of divergence critical load, and v_* is an eigenfunction of the adjoint eigenvalue problem at the point p_*.

To increase current value of the functional p_* we choose the variation $\delta m(x)$ so that $\delta p_* > 0$. Besides, new mass distribution must satisfy the condition $\int_0^1 (m(x) + \delta m(x))dx = 1$, which means that the total mass of the column remains constant. On each step of the optimization process the critical load is either $p_* = p_*^{stat}$ or $p_* = p_*^{dyn}$, and it is necessary to use the appropriate gradient function when constructing an improving variation of mass distribution. It is easy to show that the variation

$$\delta m = \alpha(x)(g(x) - \bar{g}), \ \bar{g} = \int_0^1 \alpha(x) g(x) dx \Big/ \int_0^1 \alpha(x) dx, \tag{6}$$

where $\alpha(x)$ is arbitrary nonnegative function, causes monotonous growth of the functional p_* if $\|\alpha\| << 1$, and satisfies the isoperimetric condition.

Each iteration of the optimization procedure consists of three steps. First for current mass distribution we find critical load p_* and establish the mechanism of instability: flutter or divergence. For that we solve the main and the adjoint eigenvalue problems for different values of the load parameter p. Then we calculate the appropriate gradient function $g_f(x)$ or $g_d(x)$ at the critical load p_* according to the expressions (4) or (5), respectively. On the last step we obtain the variation of the mass distribution by (6) and calculate new mass distribution

$$m^{k+1} = m^k + \delta m, \ k = 0,1,2,..$$

Consider now optimization problem (3). Since feasible control function $\mu(x)$ in (3) is bounded, Pontryagin's maximum principle can be applied. It is shown with the use of this principle that the optimal nonstructural mass distributions belong to the "bang-bang" type and, hence, have discontinuities. Finding an improving variation of nonstructural mass $\mu(x)$ reduces in this case to solving a linear programming problem.

For stability analysis of the nonconservative problems a variational principle is presented. In combination with Bubnov-Galerkin method this variational principle is used for discretization of continuous eigenvalue problems.

Solving optimization problem (2) we started with the mass distribution $m(x) \equiv 1$ (uniform column), corresponding to the critical follower load $p_* = 109.69$ destabilizing the system by flutter [1], [2]. The obtained mass distribution of the column, corresponding to the critical flutter load $p_* = 290$, is shown in Fig. 1.

Two optimal solutions of problem (3), when the mass of the payload is 10 percent of the mass of the column, differing only by upper constraint μ_{up} implied on nonstructural mass distribution, are presented in Fig. 2. It can be seen from Fig. 2a that at low values of μ_{up} optimal payload distribution has four switching points. Higher values of μ_{up} mean that a payload can be distributed more compactly along the column. Fig. 2b shows that the optimal solution tends to a Dirac delta-function as upper constraint takes higher values. Thus, a problem of optimal displacement of a concentrated mass along a uniform column ([3], [4]) naturally arises.

If we take $\mu(x) = \delta(x - a)$, then there are only three parameters in problem (3): critical end force, mass of payload κ and its displacement along the beam a. When parameter κ is fixed optimization problem reduces to finding a maximum of the critical load $p_*(a)$ as a function of displacement a of a concentrated mass. Stability diagram in the plane (a, p) when $\kappa = 0.2$ is shown in Fig. 3. It can be seen that the stability boundary has singularities and the optimal solution in this case is attained at one of the singularities (the cusp). This optimal solution corresponds to a triple real eigenvalue with the Jordan chain of dimension 3. In section 5 we will show that such singularity is typical for two-parametric circulatory systems.

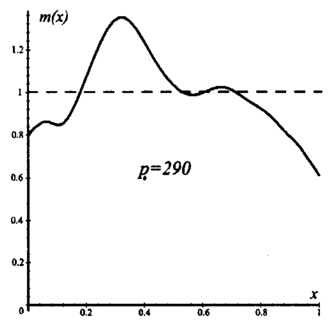

Figure 1.
Mass distribution for problem (2)

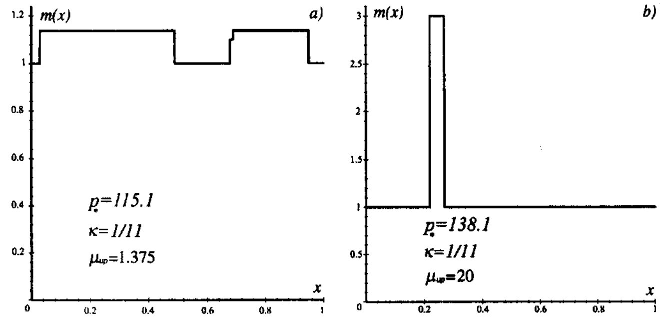

Figure 2.
Two optimal mass distributions for problem (3)

5. Singularities arising on stability boundaries in two-parametric circulatory systems

Some difficulties appear when we use gradient method for optimization of nonconservative systems. Note that the critical load functional defines the boundary between stable and unstable domains in the parameter space. During iteration process we must be able to judge about optimality of the solution obtained using the necessary optimality conditions. To obtain necessary optimality conditions and, if iteration process converges to a non-optimal solution, to construct an improving variation we must know the geometric properties of stability boundaries. In the literature on nonconservative stability problems there are many examples, including the example of section 4, showing that optimal solutions very often are attained at the singular points of stability boundaries. Thus, we need to study singularities of the boundary between stability and instability domains.

We consider two-parametric linear circulatory systems of general type with finite degrees of freedom. The case of finite degrees of freedom is important because of numerical solution needs discretization of continuous problems. Equation of motion of a linear autonomous system with nonconservative positional forces is

$$M\ddot{q} + Cq = 0\,, \tag{7}$$

where M is a symmetric real positive definite $m \times m$ matrix, C is a nonsymmetric real matrix of the same order, and q is a real vector of generalized coordinates of dimension m. Dots mean differentiation with respect to time t. Using transformation $q = ue^{\nu t}$ we obtain an eigenvalue problem $Cu + \nu^2 Mu = 0$, u being an eigenvector. Introducing notation $A = M^{-1}C$, $\lambda = -\nu^2$, we obtain an eigenvalue problem for the matrix operator A

$$Au = \lambda u\,. \tag{8}$$

It is easy to see that in terms of λ system (7) is stable iff all λ are positive and semi-simple eigenvalues; if all eigenvalues λ are real and some of them negative system (7) is statically unstable (divergence). And if at least one eigenvalue λ is complex it means flutter instability. Since the matrices M, C, and therefore the matrix A are assumed to be smoothly dependent on a vector of real parameters $p = (p_1, p_2)^T$, this criterion subdivides the plane of parameters into stability, flutter and divergence domains.

The main result on singularities of boundaries of stability and instability domains can be formulated as follows:

THEOREM *In generic case boundaries between stability, flutter, and divergence domains are smooth curves corresponding either to double real eigenvalues* (α^2) *or to simple zero eigenvalues* $(\alpha = 0)$*; at some points of the boundaries singularities* $(\alpha^2\beta^2)$, (α^3), $(\alpha^2\beta(\beta = 0))$, $(\alpha^2(\alpha = 0))$ *can appear, Fig.4.*

Here $(\alpha^2\beta^2)$ means two Jordan blocks of second order of the matrix A with real eigenvalues α and β, respectively; (α^3) means Jordan block of third order with a real eigenvalue α; and $(\alpha^2\beta(\beta = 0))$ denotes Jordan block of second order with a real eigenvalue α and simple zero eigenvalue $\beta = 0$. This result is obtained with the use of bifurcation diagrams of families of real matrices, see Arnold [5] and Galin [6].

To investigate local geometric properties of boundaries we analyse bifurcations of eigenvalues on a boundary between different domains due to variation of parameters. This method is based on the perturbation theory of eigenvalues of nonsymmetric matrices depending on parameters, developed by Vishik & Lyusternik [7] and Seyranian [8].

Taking a variation of the vector of parameters $p = p_0 + \varepsilon e + \varepsilon^2 d$, where $\varepsilon > 0$ is a small number, e and d are real vectors of the variation of dimension 2, the variation of the matrix A results:

$$A(p_0 + \varepsilon e + \varepsilon^2 d) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \dots \tag{9}$$

The matrices A_0, A_1, A_2 are described by the relations

$$A_0 = A(p_0),\ A_1 = \sum_{s=1}^{2} \frac{\partial A}{\partial p_s} e_s,\ A_2 = \sum_{i=1}^{2} \frac{\partial A}{\partial p_i} d_i + \frac{1}{2} \sum_{s,t=1}^{2} \frac{\partial^2 A}{\partial p_s \partial p_t} e_s e_t. \tag{10}$$

Due to the variation of the vector p an eigenvalue λ and an eigenvector u take increments. According to [7], these increments can be expressed as series of integer or fractional powers of ε, depending on Jordan structure corresponding to the eigenvalue λ_0. In the case of a Jordan block expansions for eigenvalues and eigenvectors contain terms with fractional powers $\varepsilon^{j/l}$, $j = 0, 1, 2, \dots$, where l is length of Jordan chain.

Consider, for example, the case of a double real eigenvalue λ_0 with length of Jordan chain equal to 2. This means that at the point $p = p_0$, belonging to a curve α^2, there exist an eigenvector u_0 and an associated vector u_1, corresponding to λ_0 and governed by equations

$$\begin{aligned} A_0 u_0 &= \lambda_0 u_0, \\ A_0 u_1 &= \lambda_0 u_1 + u_0. \end{aligned} \tag{11}$$

For the adjoint eigenvalue problem we have

$$\begin{aligned} v_0^T A_0 &= \lambda_0 v_0^T, \\ v_1^T A_0 &= \lambda_1 v_1^T + v_0^T. \end{aligned} \tag{12}$$

According to [7] expansions for perturbed eigenvalue and eigenvector are

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/2} \lambda_1 + \varepsilon \lambda_2 + \varepsilon^{3/2} \lambda_3 + \dots \\ u &= u_0 + \varepsilon^{1/2} w_1 + \varepsilon w_2 + \varepsilon^{3/2} w_3 + \dots \end{aligned} \tag{13}$$

Substituting (13) and (9) into (8) with the use of (11), (12), and normality condition $v_0^T u_0 = 1$ we obtain the expression for determining λ_1

$$\lambda_1^2 = v_0^T A_1 u_0. \tag{14}$$

If the right hand side of (14) is not zero, it yields two different nonzero roots $\lambda_1 = \pm\sqrt{v_0^T A_1 u_0}$. Using expression (10) for the matrix A_1, equation (14) can be rewritten in the form, see Seyranian [8]

$$\lambda_1^2 = \langle f_1, e \rangle + i \langle f_2, e \rangle.$$

Here $\langle a, b \rangle = \sum_{s=1}^{2} a_s b_s$ denotes inner product of vectors a, $b \in R^2$ in the parameter space. Components of the vectors f_1 and f_2 are defined by

$$f_1^s + i f_2^s = v_0^T \frac{\partial A}{\partial p_s} u_0,\ s = 1,2.$$

Hence, bifurcation of a double real eigenvalue λ_0 with Jordan chain of second order is described by the expression

$$\lambda = \lambda_0 \pm \sqrt{\langle f_1, e \rangle \varepsilon} + O(\varepsilon).$$

It is easy to see that if $\lambda_0 > 0$ and remaining eigenvalues are simple and positive, then the curve α^2 is the boundary between stability and flutter domains, and the vector f_1 is the normal vector to this boundary lying in the stability domain. If $\lambda_0 < 0$ then the curve α^2 is the boundary between divergence and flutter domains. Note that the normal vector is determined through the right and left eigenvectors, corresponding to λ_0, and first derivatives of the matrix A with respect to parameters. By the same way we find that the curve $(\alpha = 0)$ is the boundary between stability and divergence domains.

For the following presentation we need to introduce a concept of *tangent cone*, see Levantovsky [9]. Tangent cone to a set Z at its boundary point is a set of direction vectors of the curves starting at this point and lying in the set Z. A tangent cone is *nondegenerate*, if it cuts out on a sphere a set of nonzero measure. Otherwise, the cone is called *degenerate*.

Using the perturbation technique we can find tangent cones at the singular points, listed in the Theorem. At the points $\alpha^2\beta^2$ and $\alpha^2\beta(\beta=0)$ tangent cones to stability or divergence boundaries are nondegenerate:

$$\alpha^2\beta^2\text{:}\ K_{S,D}=\left\{e\in R^2\text{: } \langle f_1^a,e\rangle>0\cap\langle f_1^b,e\rangle>0\right\},$$

$$\alpha^2\beta(\beta=0)\text{:}\qquad K_S=\left\{e\text{: } \langle f_1,e\rangle>0\cap\langle g_1,e\rangle>0\right\}.$$

This means that the boundaries have discontinuities of the slope at these points. At the points α^3 and $\alpha^2(\alpha=0)$ boundaries possess cusps, and corresponding tangent cones are degenerate:

$$\alpha^3\text{:}\qquad K_{S,D}=\left\{e\text{: } \langle q_1,e\rangle=0\cap\langle r_1,e\rangle>0\right\},$$

$$\alpha^2(\alpha=0)\text{: } K_S=\left\{e\text{: } \langle f_1,e\rangle=0\cap\langle h_1,e\rangle<0\right\}.$$

All the vectors f_1, g_1, q_1, r_1, h_1 are determined by the right and left eigenvectors and associated vectors and by first derivatives of the matrix A with respect to parameters. In Fig. 4 a hypothetical division of the plane of parameters to stability, flutter, and divergence domains is presented where all types of singularities mentioned above are shown.

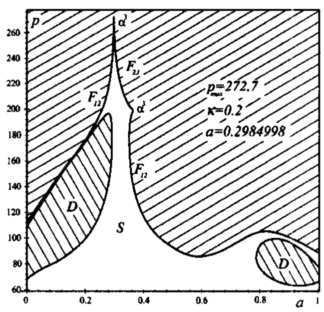

Figure 3.
Optimal displacement
of concentrated mass along the column

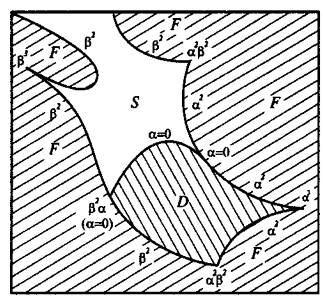

Figure 4.
Hypothetical division
of the plane of parameters

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