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MECHANICS

Effect of Small Dissipative and Gyroscopic Forces on the Stability of Nonconservative Systems

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We analyze the effect of small forces proportional to the generalized velocity vector on the stability of a linear autonomous mechanical system with nonconservative positional forces. It is known that arbitrarily small dissipation generally destabilizes a nonconservative system [1–5]. Necessary and sufficient conditions on the matrix of dissipative and gyroscopic forces under which the system is asymptotically stable are obtained. The two-dimensional system is studied in detail. The problem of the stability of the Ziegler–Herrmann–Jong pendulum is considered as a mechanical example.

1. We consider a linear mechanical system with nonconservative positional forces and small forces proportional to the velocity vector:

$$\mathbf{M}\ddot{\mathbf{q}} + \varepsilon\mathbf{D}\dot{\mathbf{q}} + \mathbf{A}\mathbf{q} = 0, \quad (1)$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$ are constant real $m \times m$ matrices determining inertial, dissipative, and gyroscopic along with nonconservative positional forces, respectively; $\varepsilon \geq 0$ is the small parameter, $\mathbf{q}$ is the generalized coordinate vector, and the dot denotes the time differentiation. The matrix $\mathbf{M}$ is assumed nonsingular.

Substituting $\mathbf{q} = \mathbf{u}e^{\lambda t}$, we arrive at the eigenvalue problem

$$(\mathbf{M}\lambda^2 + \varepsilon\mathbf{D}\lambda + \mathbf{A})\mathbf{u} = 0. \quad (2)$$

Eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{2m}$ are determined from the characteristic equation

$$\det(\mathbf{M}\lambda^2 + \varepsilon\mathbf{D}\lambda + \mathbf{A}) = 0. \quad (3)$$

We now consider system (1) in the absence of forces proportional to the velocity vector ($\varepsilon = 0$). This system is called the circulatory system [1, 2]. In this case, as follows from Eq. (3), if λ is an eigenvalue, $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$ are also eigenvalues. Therefore, the circulatory system is stable (not asymptotically) if and only if all eigenvalues $\pm i\omega_j$, $\omega_j \geq 0_j$ are imaginary and semisimple. This means that the number r of independent eigenvectors corresponding to an eigenvalue is equal to its algebraic multiplicity k. When $r < k$, the general solution of system (1) contains secular terms proportional to $t^{\alpha}e^{\lambda t}$, $\alpha \leq k - 1$ (instability). Thus, the system having a pair of algebraically double eigenvalues $\pm i\omega_0$, $\omega_0 > 0$ with one eigenvector, where other eigenvalues are imaginary and simple, corresponds to the boundary between the regions of stability and instability (flutter) [6]. Let us analyze this case in more detail.

The right ($\mathbf{u}_0$, $\mathbf{u}_1$) and left ($\mathbf{v}_0$, $\mathbf{v}_1$) eigenvectors and adjoint vectors corresponding to the double eigenvalue $\lambda_0 = i\omega_0$ are determined from the equations [7, 8]

$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{u}_0 = 0,$$
$$(\mathbf{A} - \omega_0^2\mathbf{M})\mathbf{u}_1 = -2i\omega_0\mathbf{M}\mathbf{u}_0, \quad (4)$$

$$\mathbf{v}_0^{\mathrm{T}}(\mathbf{A} - \omega_0^2\mathbf{M}) = 0,$$
$$\mathbf{v}_1^{\mathrm{T}}(\mathbf{A} - \omega_0^2\mathbf{M}) = -2i\omega_0\mathbf{v}_0^{\mathrm{T}}\mathbf{M}. \quad (5)$$

In addition, they are related as

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_0 = 0, \quad \mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1 = \mathbf{v}_1^{\mathrm{T}}\mathbf{M}\mathbf{u}_0 \neq 0. \quad (6)$$

The vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, and $\mathbf{v}_1$ are defined up to arbitrary constants. Since the matrix $\mathbf{A} - \omega_0^2\mathbf{M}$ is real, the eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$ in Eqs. (4) and (5) can be taken real. In this case, the adjoint vectors $\mathbf{u}_1$ and $\mathbf{v}_1$ are imaginary.

In the presence of small dissipative and gyroscopic forces ($\varepsilon > 0$), the double eigenvalue $\lambda_0 = i\omega_0$ with one eigenvector generally splits into two simple eigenvalues. This splitting is determined by the expansion

$$\lambda = i\omega_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \ldots, \quad (7)$$

where the coefficient λ_1 is determined from the qua-

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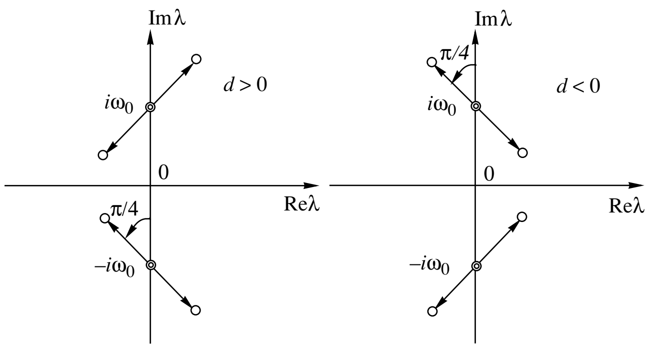

Fig. 1. Destabilization of the circulatory system by the small perturbation ε**D**.

dratic equation [7, 8]

$$\lambda_1^2 = id, \quad d = -\frac{\mathbf{v}_0^T \mathbf{D} \mathbf{u}_0}{2i\mathbf{v}_0^T \mathbf{M} \mathbf{u}_1}. \quad (8)$$

We note that the quantity d is real, because the vectors $\mathbf{u}_0$ and $\mathbf{v}_0$ are real, while the vector $\mathbf{u}_1$ is imaginary. Therefore, in the presence of perturbation $\varepsilon\mathbf{D}$ ($\varepsilon > 0$), the double eigenvalue $\lambda_0 = i\omega_0$ splits into two simple eigenvalues

$$\lambda = i\omega_0 \pm \sqrt{id\varepsilon} + O(\varepsilon).$$

For $d \neq 0$, these eigenvalues lie on the opposite sides of the imaginary axis (Fig. 1). This means destabilization of the circulatory system ($\varepsilon = 0$) by arbitrarily small forces proportional to the velocity vector.

Therefore, $d = 0$, i.e.,

$$\frac{\mathbf{v}_0^T \mathbf{D} \mathbf{u}_0}{2i\mathbf{v}_0^T \mathbf{M} \mathbf{u}_1} = 0 \quad (9)$$

is a necessary condition of stabilization of the system. Under this condition, splitting of the double eigenvalue is determined by the expansion $\lambda = i\omega_0 + \lambda_2\varepsilon + o(\varepsilon)$, where the coefficient λ_2 is determined from the quadratic equation [7]

$$\lambda_2^2 + \lambda_2 \frac{\mathbf{v}_1^T \mathbf{D} \mathbf{u}_0 + \mathbf{v}_0^T \mathbf{D} \mathbf{u}_1}{2\mathbf{v}_0^T \mathbf{M} \mathbf{u}_1} - i\omega_0 \frac{\mathbf{v}_0^T \mathbf{D} \mathbf{G}(\mathbf{D} \mathbf{u}_0)}{2\mathbf{v}_0^T \mathbf{M} \mathbf{u}_1} = 0. \quad (10)$$

Here, $\mathbf{G}$ is the operator inverse to the operator $\mathbf{A} - \omega_0^2 \mathbf{M}$. In particular, this operator can be represented in the form

$$\mathbf{G} = (\mathbf{A} - \omega_0^2 \mathbf{M} + 2i\omega_0 \mathbf{v}_0 \mathbf{v}_1^T \mathbf{M})^{-1}, \quad \det \mathbf{G} \neq 0.$$

The coefficients of Eq. (10) are real. If the circulatory system is stabilized by small forces proportional to the velocity vector, both roots λ_2 must satisfy the condition $\mathrm{Re}\lambda_2 \leq 0$. This condition is equivalent to the weakened Routh–Hurwitz conditions for polynomial (10):

$$\frac{\mathbf{v}_1^T \mathbf{D} \mathbf{u}_0 + \mathbf{v}_0^T \mathbf{D} \mathbf{u}_1}{2\mathbf{v}_0^T \mathbf{M} \mathbf{u}_1} \geq 0, \quad (11)$$

$$-i\omega_0 \frac{\mathbf{v}_0^T \mathbf{D} \mathbf{G}(\mathbf{D} \mathbf{u}_0)}{2\mathbf{v}_0^T \mathbf{M} \mathbf{u}_1} \geq 0. \quad (12)$$

Strict inequalities (11) and (12) for sufficiently small values $\varepsilon > 0$ ensure splitting of the double eigenvalue $\lambda_0 = i\omega_0$ into two eigenvalues lying in the left half-plane.

In addition to a double pair $\pm i\omega_0$, the behavior of simple eigenvalues $\pm i\omega_j$, $j = 3, 4, \ldots, m$ with the right $\mathbf{u}_j$ and left $\mathbf{v}_j$ eigenvectors must be studied. When introducing small dissipative and gyroscopic forces ($\varepsilon > 0$), increments of these eigenvalues are determined by the formula

$$\lambda_j = i\omega_j + \mu_j \varepsilon + O(\varepsilon^2),$$

where

$$\mu_j = -\frac{\mathbf{v}_j^T \mathbf{D} \mathbf{u}_j}{2\mathbf{v}_j^T \mathbf{M} \mathbf{u}_j}$$

are real. Thus, the conditions

$$\frac{\mathbf{v}_j^T \mathbf{D} \mathbf{u}_j}{\mathbf{v}_j^T \mathbf{M} \mathbf{u}_j} \geq 0, \quad j = 3, 4, \ldots, m \quad (13)$$

in the first approximation in ε mean that any simple eigenvalue λ_j does not transit to the right half-plane in the presence of the perturbation $\varepsilon\mathbf{D}$ ($\varepsilon > 0$). Strict inequalities (13) imply that perturbed eigenvalues λ_j belong to the left half-plane for sufficiently small $\varepsilon > 0$.

Conditions (9) and (11)–(13) are the constructive necessary conditions of stabilization of the circulatory system by small dissipative and gyroscopic forces. Correspondingly, sufficient conditions of stabilization of system (1) are derived from conditions (9) and (11)–(13) by replacing nonstrict inequalities with strict ones. These conditions impose constraints on the elements of the matrix **D**. Conditions (9), (11), and (13) are linear, and condition (12) is quadratic in the elements of the matrix **D**. To calculate the coefficients of linear and quadratic forms, it is necessary to know the spectrum of the circulatory system and corresponding right and left eigenvectors and adjoint vectors. We emphasize that one constraint specified by equality (9) and m constraints specified by inequalities (11)–(13) are imposed on the m^2 elements of the matrix $\mathbf{D}$.

2. We consider system (1) under the assumption that $m = 2$ and $\mathbf{M} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. This assumption does not limit generality, and the results of this section can be extended to the case of the arbitrary mass matrix with $\det\mathbf{M} \neq 0$. The spectrum of the two-dimensional system at the boundary between regions of stability and flutter consists of only a pair of imaginary double eigenvalues $\pm i\omega_0$. Since simple eigenvalues are absent, the stability of the system is determined by the behavior of this pair.

The necessary and sufficient condition of the existence of the double eigenvalue $\lambda_0 = i\omega_0$ of the circulatory system can be represented as the equation

$$4a_{12}a_{21} + (a_{22} - a_{11})^2 = 0, \qquad (14)$$

equivalent to the equality $\det\mathbf{A} = \left(\frac{\mathrm{tr}\mathbf{A}}{2}\right)^2$. In this case,

$$-\lambda_0^2 = \omega_0^2 = \frac{a_{11} + a_{22}}{2} > 0, \quad a_{12}a_{21} \le 0. \qquad (15)$$

In view of conditions (14) and (15), the eigenvectors and adjoint vectors $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{u}_1$, and $\mathbf{v}_1$ corresponding to the double eigenvalue $\lambda_0 = i\omega_0$ are found from Eqs. (4) and (5) in the form

$$\mathbf{u}_0 = \begin{bmatrix} 2a_{12} \\ a_{22} - a_{11} \end{bmatrix}, \quad \mathbf{v}_0 = \begin{bmatrix} 2a_{21} \\ a_{22} - a_{11} \end{bmatrix}, \qquad (16)$$

$$\mathbf{u}_1 = \begin{bmatrix} 0 \\ -4i\omega_0 \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 0 \\ -4i\omega_0 \end{bmatrix}. \qquad (17)$$

Therefore, the denominator of expressions (8) and (10) is equal to

$$2\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1 = -8i\omega_0(a_{22} - a_{11}). \qquad (18)$$

In view of Eqs. (18) and (14), necessary condition (9) takes the form

$$\frac{\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_0}{2i(\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1)} = \frac{(d_{22} - d_{11})(a_{22} - a_{11}) + 2(d_{12}a_{21} + d_{21}a_{12})}{8\omega_0} = 0. \qquad (19)$$

This condition can be written in the compact form

$$2\mathrm{tr}(\mathbf{AD}) = \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D}.$$

We now determine the coefficients of quadratic equation (10). The coefficient of the linear term is equal to

$$\mathbf{v}_1^{\mathrm{T}}\mathbf{D}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{u}_1 = -8i\omega_0(d_{22}(a_{22} - a_{11}) + d_{12}a_{21} + d_{21}a_{12}) = -4i\omega_0\mathrm{tr}\mathbf{D}(a_{22} - a_{11}). \qquad (20)$$

To determine the free term of Eq. (10), the vector $\mathbf{w}$ must be determined from the inhomogeneous equation

$$(\mathbf{A} - \omega_0^2\mathbf{I})\mathbf{w} = \mathbf{D}\mathbf{u}_0, \qquad (21)$$

where the eigenvector $\mathbf{u}_0$ is given in Eqs. (16). Solving Eq. (21), we obtain

$$\mathbf{G}(\mathbf{D}\mathbf{u}_0) \equiv \mathbf{w} = \begin{bmatrix} -2d_{12} \\ 2d_{11} \end{bmatrix}. \qquad (22)$$

Then,

$$\mathbf{v}_0^{\mathrm{T}}\mathbf{D}\mathbf{G}(\mathbf{D}\mathbf{u}_0) = 2(a_{22} - a_{11})\det\mathbf{D}. \qquad (23)$$

Substituting Eqs. (18), (20), and (23) into quadratic equation (10), we arrive at the relation

$$\lambda_2^2 + \lambda_2\frac{1}{2}\mathrm{tr}\mathbf{D} + \frac{1}{4}\det\mathbf{D} = 0. \qquad (24)$$

Thus, necessary conditions (9), (11), and (12) for the two-dimensional system ($m = 2$) take the compact form

$$2\mathrm{tr}(\mathbf{AD}) = \mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D}, \qquad (25)$$

$$\mathrm{tr}\mathbf{D} \ge 0, \quad \det\mathbf{D} \ge 0. \qquad (26)$$

We note that similar conditions and Eq. (24) were obtained in [5] by analyzing the characteristic polynomial of system (1).

Let us determine the stabilization region that is specified by strict conditions (25) and (26) in the space of the elements of the matrix $\mathbf{D}$. Two cases are naturally distinguished.

In the first case, where $a_{12} \neq 0$, d_{21} is expressed from equality (25), and the matrix of dissipative and gyroscopic forces is found in the form

$$\mathbf{D} = \begin{bmatrix} d_{11} & d_{12} \\ \dfrac{(d_{22} - d_{11})(a_{11} - a_{22}) - 2a_{21}d_{12}}{2a_{12}} & d_{22} \end{bmatrix}. \qquad (27)$$

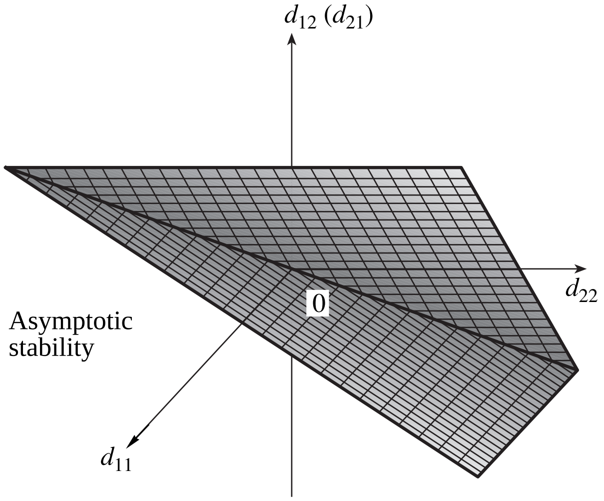

Fig. 2. Stabilization region for the asymmetric matrix **D** for $\frac{a_{11}-a_{22}}{2a_{12}} > 0$ $\left(\text{or } \frac{a_{11}-a_{22}}{2a_{21}} > 0\right)$.

Using condition (14) that the eigenvalue λ_0 is double, we transform inequalities (26) for matrix (27) to the form

$$d_{11} + d_{22} \geq 0, \tag{28}$$

$$\left(d_{11} - d_{12}\frac{a_{11}-a_{22}}{2a_{12}}\right)\left(d_{22} - d_{12}\frac{a_{22}-a_{11}}{2a_{12}}\right) \geq 0. \tag{29}$$

These inequalities are equivalent to the conditions

$$d_{11} \geq d_{12}\frac{a_{11}-a_{22}}{2a_{12}}, \quad d_{22} \geq d_{12}\frac{a_{22}-a_{11}}{2a_{12}}. \tag{30}$$

Thus, in the three-dimensional space of the parameters d_{11}, d_{22}, and d_{12}, inequalities (30) define the dihedral angle that determine the region of stabilization of the circulatory system by small dissipative and gyroscopic forces specified by matrix **D** (27) (Fig. 2).

In the second case, where $a_{21} \neq 0$, the necessary conditions of stability have the form

$$d_{11} \geq d_{21}\frac{a_{11}-a_{22}}{2a_{21}}, \quad d_{22} \geq d_{21}\frac{a_{22}-a_{11}}{2a_{21}}, \tag{31}$$

which corresponds to the dihedral angle in the three-dimensional space of the parameters d_{11}, d_{22}, and d_{21}. In this case, the matrix **D** has the form

$$\mathbf{D} = \begin{bmatrix} d_{11} & \frac{(d_{22}-d_{11})(a_{11}-a_{22}) - 2a_{12}d_{21}}{2a_{21}} \\ d_{21} & d_{22} \end{bmatrix}. \tag{32}$$

When $a_{12} \neq 0$ and $a_{21} \neq 0$, conditions (30) and (31) corresponding to matrices (27) and (30), respectively, are equivalent to each other.

Let us consider the case where $a_{11} = a_{22}$. In this case, $a_{12} = 0$ or $a_{21} = 0$. These equalities cannot be satisfied simultaneously. Otherwise, two linearly independent eigenvectors would correspond to the double eigenvalue λ_0 that contradicts the initial assumption. It follows from condition (25) that $d_{12} = 0$ or $d_{21} = 0$, respectively. According to strict conditions (30) and (31), the stabilization region in the three-dimensional space of the parameters d_{11}, d_{22}, and d_{12} (or d_{21}) is the right dihedral angle specified by the inequalities $d_{11} > 0$ and $d_{22} > 0$.

We now determine the form of the symmetric matrices **D** stabilizing the circulatory system. In this case, gyroscopic forces are absent, and inequalities (26) mean that the matrix **D** is nonnegative. We emphasize that strict inequalities (26) imply total dissipation. Expressing the coefficient $d_{12} = d_{21}$ from Eq. (25), we obtain

$$\mathbf{D} = \begin{bmatrix} d_{11} & \frac{(a_{22}-a_{11})(d_{11}-d_{22})}{2(a_{12}+a_{21})} \\ \frac{(a_{22}-a_{11})(d_{11}-d_{22})}{2(a_{12}+a_{21})} & d_{22} \end{bmatrix}. \tag{33}$$

We note that $a_{12} + a_{21} \neq 0$, because otherwise two linearly independent eigenvectors would correspond to the double eigenvalue λ_0 that contradicts the initial assumption. Calculating the determinant and trace of matrix (33) and writing the conditions that they are nonnegative, we arrive at the following necessary conditions in the space of the two parameters d_{11} and d_{22}:

$$d_{11}, d_{22} \geq 0, \quad \frac{\sqrt{x}-1}{\sqrt{x}+1}d_{22} \leq d_{11} \leq d_{22}\frac{\sqrt{x}+1}{\sqrt{x}-1},$$
$$x = 1 + \left(\frac{a_{22}-a_{11}}{a_{12}+a_{21}}\right)^2. \tag{34}$$

Strict inequalities (34) specify the region of stabilization of system (1) by small forces proportional to the velocity vector. Thus, the region of stabilization of the circulatory system by symmetric matrices $\varepsilon\mathbf{D}$ of the form specified by Eqs. (33) and (34) is an angle on the plane of parameters d_{11} and d_{22} (Fig. 3). According to formula (34), this angle is generally acute and is right only for $a_{11} = a_{22}$.

3. Let us consider the Ziegler–Herrmann–Jong pendulum [1, 9] consisting of two rigid massless bars, which have the same length l and are connected by a hinge, and point-like masses $m_1 = 2m$ and $m_2 = m$ located at the bar connection point and free end, respectively (Fig. 4). The pendulum is subject to the follower

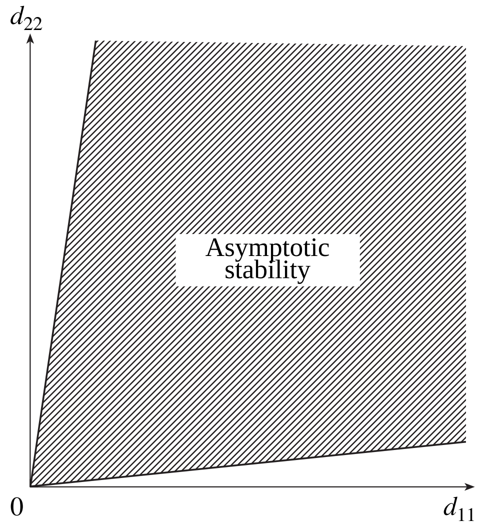

Fig. 3. Stabilization region for the symmetric matrix **D**.

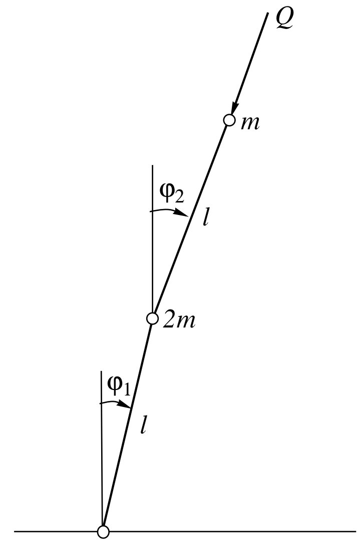

Fig. 4. Ziegler–Herrmann–Jong pendulum.

force Q applied to the free end. The viscoelastic hinges of the pendulum have the same rigidity c and different damping coefficients εb_1 and εb_2. In terms of the dimensionless quantities

$$q = \frac{Ql}{c}, \quad k_1 = \frac{b_1}{\sqrt{cml^2}}, \quad k_2 = \frac{b_2}{\sqrt{cml^2}},$$
$$\tau = t\sqrt{\frac{c}{ml^2}},$$

where τ is the time, the equations of small oscillations of the pendulum have the form

$$\frac{d^2\mathbf{y}}{d\tau^2} + \varepsilon\mathbf{D}\frac{d\mathbf{y}}{d\tau} + \mathbf{A}\mathbf{y} = 0, \quad \mathbf{y} = \begin{bmatrix}\varphi_1\\ \varphi_2\end{bmatrix}, \tag{35}$$

where

$$\mathbf{D} = \frac{1}{2}\begin{bmatrix}k_1 + 2k_2 & -2k_2\\ -k_1 - 4k_2 & 4k_2\end{bmatrix}, \quad \mathbf{A} = \frac{1}{2}\begin{bmatrix}3-q & q-2\\ q-5 & 4-q\end{bmatrix}. \tag{36}$$

It is known that, in the absence of viscous friction, when $\varepsilon = 0$, the equilibrium position of the pendulum is stable for $q < q_0 = \frac{7}{2} - \sqrt{2}$ [9]. The critical load q_0 corresponds to the boundary between the regions of stability and flutter of the circulatory system. At this point, the spectrum of the system includes a pair of double imaginary eigenvalues $\pm i\omega_0$ and $\omega_0 = 2^{-1/4}$ with one eigenvector.

To determine the damping parameters k_1 and k_2 for which the perturbed system is asymptotically stable, we use stabilization conditions (25) and (26). Calculating the invariants of the matrices **A** and **D** for $q = q_0$

$$\mathrm{tr}\mathbf{A} = \sqrt{2}, \quad \mathrm{tr}\mathbf{D} = \frac{1}{2}k_1 + 3k_2, \quad \det\mathbf{D} = \frac{1}{2}k_1k_2,$$
$$\mathrm{tr}(\mathbf{AD}) = \left(-\frac{1}{2} + \frac{\sqrt{2}}{2}\right)k_1 + \left(-\frac{1}{2} + 3\sqrt{2}\right)k_2 \tag{37}$$

and substituting them into relations (25) and (26), we arrive at the necessary conditions of stabilization

$$k_1 = (5\sqrt{2} + 4)k_2, \quad k_2 \geq 0. \tag{38}$$

Thus, if the damping coefficients in the hinges satisfy strict conditions (38), the Ziegler–Herrmann–Jong pendulum is asymptotically stable.

We now determine the general form of the matrix **D** stabilizing circulatory system (1) without constrains (36). Substituting the coefficients of the matrix **A** calculated at the critical point $q = q_0$ into formulas (31) and (32), we obtain

$$\mathbf{D} = \begin{bmatrix}d_{11} & (17 - 12\sqrt{2})d_{21} + (3 - 2\sqrt{2})(d_{22} - d_{11})\\ d_{21} & d_{22}\end{bmatrix} \tag{39}$$

with the constrains

$$-d_{22} \leq d_{21}(3 - 2\sqrt{2}) \leq d_{11}. \tag{40}$$

for the coefficients. It is easy to check that if the matrix **D** has form (36), conditions (39) and (40) lead to relations (38).

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