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Destabilization Paradox

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In 1952, analyzing the stability of a two-link pendulum loaded by a follower force, H. Ziegler [1] surprisingly concluded that the critical force at which a nonconservative system with negligibly low dissipation lost stability was much weaker than that in a system where dissipation was absent from the very beginning. This phenomenon, called the destabilization paradox, was later found in many mechanical and physical systems [2–4]. Despite numerous works, problems generated by the destabilization paradox have not yet been generally solved, although they are of the most theoretical interest according to Bolotin [2]. In this work, a theory is developed to both qualitatively and quantitatively explain the paradoxical behavior of general nonconservative systems under the action of weak dissipative and gyroscopic forces. The problem of the stability of the Reut–Sugiyama pendulum is analyzed as an example.

1. Let us consider a linear autonomous nonconservative mechanical system described by the equation

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{D}(\mathbf{k})\dot{\mathbf{y}} + \mathbf{A}(q)\mathbf{y} = 0, \qquad (1)$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$ are the real $m \times m$ matrices specifying the inertial, dissipative along with gyroscopic, and nonconservative position forces, respectively; $\mathbf{y}$ is the generalized coordinate vector; and the dots stand for differentiation with respect to time t. The matrix $\mathbf{D}$ is a smooth function of the parameter vector $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$, $\mathbf{D}(0) = 0$, the matrix $\mathbf{A}$ is a smooth function of the scalar load parameter $q \geq 0$, and the matrix $\mathbf{M}$ is parametrically independent.

Seeking a solution of Eq. (1) in the form $\mathbf{y} = \mathbf{u}\exp(\lambda t)$, we arrive at the generalized eigenvalue problem

$$(\lambda^2\mathbf{M} + \lambda\mathbf{D}(\mathbf{k}) + \mathbf{A}(q))\mathbf{u} = 0, \qquad (2)$$

where $\mathbf{u}$ is the eigenvector and λ is the eigenvalue. A nonconservative system without gyroscopic and dissipative forces ($\mathbf{k} = 0$), which is described by the equation

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{A}(q)\mathbf{y} = 0 \qquad (3)$$

is called the circulatory system [1–4]. The spectrum of the circulatory system is mirror symmetric; i.e., if λ is an eigenvalue of the linear operator $\lambda^2\mathbf{M} + \mathbf{A}(q)$, then $-\lambda$, $\bar{\lambda}$, and $-\bar{\lambda}$, where the bar stands for complex conjugation, are also eigenvalues. Therefore, the circulatory system is stable in the Lyapunov sense if all its eigenvalues λ are imaginary and semisimple [5].

Let the circulatory system be stable for $q = 0$. When the load parameter increases and reaches a certain critical value $q = q_0$, two simple imaginary eigenvalues can merge into a double eigenvalue $i\omega_0$ with the Jordan chain of a length of 2. With further increase in the load, the double eigenvalue generally splits into a pair of complex eigenvalues; one of them has a positive real part, which means vibration instability (flutter, Fig. 1a). Thus, the range $0 \leq q < q_0$ belongs to the stability region of the unperturbed system described by Eq. (3) [5].

Perturbation of the circulatory system by weak dissipative and gyroscopic forces ($\mathbf{k} \neq 0$) breaks the coupling between the eigenvalues and, when the load parameter reaches a certain critical value $q = q_{cr}(k_1, k_2, ..., k_{n-1})$, leads to the displacement of one of the eigenvalues to the right-hand side of the complex plane without the formation and further bifurcation of the double eigenvalue (Fig. 1a). Moreover, if $\mathbf{k} = \varepsilon\tilde{\mathbf{k}}$, where $\tilde{\mathbf{k}}$ is the fixed vector and $\varepsilon \longrightarrow 0$,

$$\tilde{q}_{cr} \equiv \lim_{\varepsilon \to 0} q_{cr}(\varepsilon\tilde{\mathbf{k}}) \leq q_0. \qquad (4)$$

This inequality expresses the destabilization paradox first pointed out in [1]: the critical load can abruptly decrease when infinitely weak gyroscopic and dissipative forces are taken into account. More recently, for various mechanical systems, it was shown that the limiting critical load $\tilde{q}_{cr}$ depends on the choice of the vector $\tilde{\mathbf{k}}$ [2–4]. In particular, changing the relation between the parameters $k_1, k_2, \ldots, k_{n-1}$, one can avoid a decrease in the critical load and thereby destabilization (Bolotin effect [2]). For the two-dimensional Ziegler pendulum with two dissipation parameters, Seyranian [7] found a region on the parameter plane where a nonconservative system perturbed by weak dissipative forces was asymptotically stable and $q_{cr}(\mathbf{k}) > q_0$.

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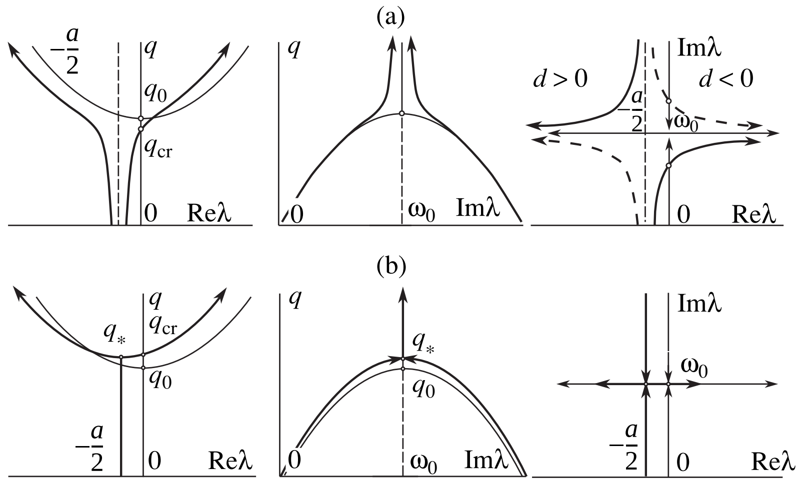

Fig. 1. Trajectories of the eigenvalues of the (thin lines) unperturbed circulatory system and (thick lines) system with weak velocity-dependent forces for (a) $d \neq 0$ and (b) $d = 0$.

In this work, for the general linear nonconservative system described by Eq. (1), an explicit approximation is found for the function $q_{cr}(\mathbf{k})$, which makes it possible to determine both the jump of the critical load and the asymptotic-stability region. In addition, explicit asymptotic expressions are obtained for the description of the trajectories of eigenvalues and their decomposition into independent curves under perturbations of the circulatory system by weak dissipative and gyroscopic forces.

2. Let us consider the point $\mathbf{p}_0 = (0, \dots, 0, q_0)$ in the n-dimensional space of the parameters $k_1, k_2, \dots, k_{n-1}$, and q of the system described by Eq. (1). It is assumed that $\pm i\omega_0$, where $\omega_0 > 0$, are the double eigenvalues of the operator $\mathbf{A}(q_0) + \lambda^2\mathbf{M}$ with the Jordan chain of a length of 2 and the remaining eigenvalues $\pm i\omega_{0,s}$, where $\omega_{0,s} > 0$ and $s = 1, 2, \dots, m-2$, are imaginary and simple. The nonconservative system corresponding to $\mathbf{k} = 0$ and $q = q_0$ is a circulatory system, and the point $\mathbf{p}_0$ belongs to the boundary of the stability region.

Eigenvectors $\mathbf{u}_0$ and $\mathbf{v}_0$, as well as associated vectors $\mathbf{u}_1$ and $\mathbf{v}_1$, corresponding to the double eigenvalue $i\omega_0$ satisfy the equations

$$(\mathbf{A}(q_0) - \omega_0^2\mathbf{M})\mathbf{u}_0 = 0,$$
$$(\mathbf{A}(q_0) - \omega_0^2\mathbf{M})\mathbf{u}_1 = -2i\omega_0\mathbf{M}\mathbf{u}_0, \quad (5)$$

$$\mathbf{v}_0^{\mathrm{T}}(\mathbf{A}(q_0) - \omega_0^2\mathbf{M}) = 0,$$
$$\mathbf{v}_1^{\mathrm{T}}(\mathbf{A}(q_0) - \omega_0^2\mathbf{M}) = -2i\omega_0\mathbf{v}_0^{\mathrm{T}}\mathbf{M}. \quad (6)$$

The vectors $\mathbf{u}_0$, $\mathbf{v}_0$ and $\mathbf{u}_1$, $\mathbf{v}_1$ are taken to be real and imaginary, respectively, so that

$$2i\omega_0\mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1 = 1,$$
$$2i\omega_0\mathbf{v}_1^{\mathrm{T}}\mathbf{M}\mathbf{u}_1 + \mathbf{v}_1^{\mathrm{T}}\mathbf{M}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\mathbf{M}\mathbf{u}_1 = 0. \quad (7)$$

Let us analyze the stability of system (1) under the linear perturbation of the parameter vector $\mathbf{p} = (\mathbf{k}, q)$:

$$\mathbf{p}(\varepsilon) = *_0 + \varepsilon\mathbf{p}', \quad \varepsilon \geq 0, \quad (8)$$

where prime means the derivative with respect to ε at $\varepsilon = 0$. The perturbed double eigenvalue is generally expanded into the Newton–Puiseux series

$$\lambda = i\omega_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \dots, \quad (9)$$

where the coefficients λ_1 and λ_2 are determined from the equations [8]

$$\lambda_1^2 = -i\omega_0\langle\mathbf{f}, \mathbf{k}'\rangle - \tilde{f}q',$$
$$2\lambda_2 = -\langle\mathbf{f} - \omega_0\mathbf{h}, \mathbf{k}'\rangle - i\tilde{h}q'. \quad (10)$$

Here, angular brackets mean the scalar product of the vector $\mathbf{k}' = (k_1', k_2', \dots, k_{n-1}')$ and real vectors $\mathbf{f}$ and $\mathbf{h}$ with the components

$$f_r = \mathbf{v}_0^{\mathrm{T}}\frac{\partial\mathbf{D}}{\partial k_r}\mathbf{u}_0, \quad ih_r = \mathbf{v}_1^{\mathrm{T}}\frac{\partial\mathbf{D}}{\partial k_r}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\frac{\partial\mathbf{D}}{\partial k_r}\mathbf{u}_1,$$
$$r = 1, 2, \dots, n-1, \quad (11)$$

and the real values $\tilde{f}$ and $\tilde{h}$ are given by the expressions

$$\tilde{f} = \mathbf{v}_0^{\mathrm{T}}\frac{\partial\mathbf{A}}{\partial q}\mathbf{u}_0, \quad i\tilde{h} = \mathbf{v}_1^{\mathrm{T}}\frac{\partial\mathbf{A}}{\partial q}\mathbf{u}_0 + \mathbf{v}_0^{\mathrm{T}}\frac{\partial\mathbf{A}}{\partial q}\mathbf{u}_1. \quad (12)$$

Thus, from Eqs. (8)–(10), we obtain [8]

$$\lambda = i\omega_0 \pm \sqrt{-i\omega_0\langle\mathbf{f}, \mathbf{k}\rangle - \tilde{f}(q - q_0)}$$
$$-\frac{1}{2}(\langle\mathbf{f} - \omega_0\mathbf{h}, \mathbf{k}\rangle + i\tilde{h}(q - q_0)) + \dots. \quad (13)$$

If the radicand is nonzero, this equation describes the splitting of the double eigenvalue $i\omega_0$ when varying the parameters **k** and q. In this case, $i\omega_0$ splits generally into two simple eigenvalues, one of which has a positive real part (instability). If $\langle \mathbf{f}, \mathbf{k}\rangle = 0$, the square root in Eq. (13) is imaginary for $\tilde{f}(q - q_0) > 0$, and the condition $\langle \mathbf{h}, \mathbf{k}\rangle < 0$ is necessary for asymptotic stability. In this case, under sufficiently small perturbations (8), the eigenvalue $i\omega_0$ (and $-i\omega_0$) splits into two simple eigenvalues with negative real parts.

In addition, the stability of system (1) is determined by the behavior of $2m - 4$ simple eigenvalues $\pm i\omega_{0,s}$. Let us take the real right $\mathbf{u}_{0,s}$ and left $\mathbf{v}_{0,s}$ eigenvectors that correspond to the eigenvalues $i\omega_{0,s}$ and satisfy the normalization conditions

$$2\omega_{0,s}\mathbf{v}_{0,s}^{\mathrm{T}}\mathbf{M}\mathbf{u}_{0,s} = 1. \tag{14}$$

The increment of the eigenvalues $\pm i\omega_{0,s}$ under perturbations (8) has the form

$$\begin{gathered}\lambda = \pm i\omega_{0,s} \mp i\tilde{g}_s(q - q_0) - \omega_{0,s}\langle \mathbf{g}_s, \mathbf{k}\rangle + \dots, \\ s = 1, 2, \dots, m-2,\end{gathered} \tag{15}$$

where $\tilde{g}_s$ and the components of the real vector $\mathbf{g}_s$ are given by the expressions

$$\begin{gathered}\tilde{g}_s = \mathbf{v}_{0,s}^{\mathrm{T}}\frac{\partial \mathbf{A}}{\partial q}\mathbf{u}_{0,s}, \quad g_{s,r} = \mathbf{v}_{0,s}^{\mathrm{T}}\frac{\partial \mathbf{D}}{\partial k_r}\mathbf{u}_{0,s}, \\ r = 1, 2, \dots, n-1.\end{gathered} \tag{16}$$

If $\langle \mathbf{g}_s, \mathbf{k}\rangle > 0$, then $\mathrm{Re}\lambda < 0$. Thus, under the conditions

$$\begin{gathered}\langle \mathbf{f}, \mathbf{k}\rangle = 0, \quad \langle \mathbf{h}, \mathbf{k}\rangle < 0, \quad \tilde{f}(q - q_0) > 0, \\ \langle \mathbf{g}_s, \mathbf{k}\rangle > 0, \quad s = 1, 2, \dots, m-2,\end{gathered} \tag{17}$$

system (1) is asymptotically stable for sufficiently small linear variations of the parameters **k** and q.

3. According to relations (17), the set of directions from the point $\mathbf{p}_0 = (0, q_0)$ to the asymptotic-stability region has the dimension $n - 1$ in the n-dimensional parameter space. It is known that the dimension of the asymptotic-stability region coincides with the dimension of the parameter space [9]. This means that the asymptotic-stability region can be reached only along the curve touching the plane $\langle \mathbf{f}, \mathbf{k}\rangle = 0$ at the point $\mathbf{p}_0$. To gain more precise information about the shape of the asymptotic-stability region near the point $\mathbf{p}_0$, let us consider smooth variation of the parameter vector

$$\mathbf{p}(\varepsilon) = \begin{bmatrix} 0 \\ q_0 \end{bmatrix} + \varepsilon\begin{bmatrix} \mathbf{k}' \\ 0 \end{bmatrix} + \frac{\varepsilon^2}{2}\begin{bmatrix} \mathbf{k}'' \\ q'' \end{bmatrix} + o(\varepsilon^2) \tag{18}$$

under the assumption that

$$\langle \mathbf{f}, \mathbf{k}'\rangle = 0. \tag{19}$$

The curve specified by Eqs. (18) and (19) is orthogonal to the q axis in the parameter space, because $q' \equiv 0$.

The coefficient λ_1 of expansion (9) that is determined by the first of Eqs. (10) vanishes along the curve specified by Eqs. (18) and (19). Therefore, the double eigenvalue splits linearly with respect to ε:

$$\lambda = i\omega_0 + \lambda_2\varepsilon + \dots, \tag{20}$$

where the coefficient λ_2 is the root of the quadratic equation [8]

$$\begin{gathered}2\lambda_2^2 - 2\lambda_2\omega_0\langle \mathbf{h}, \mathbf{k}'\rangle + (\tilde{f}q'' + 2\omega_0^2\langle \mathbf{G}\mathbf{k}', \mathbf{k}'\rangle) \\ + i\omega_0(\langle \mathbf{f}, \mathbf{k}''\rangle + 2\langle \mathbf{H}\mathbf{k}', \mathbf{k}'\rangle) = 0.\end{gathered} \tag{21}$$

Here, the vectors **f**, **h** and quantities $\tilde{f}$, $\tilde{h}$ are determined by Eqs. (11) and (12), respectively; the real matrix **H** has the components

$$H_{rs} = \frac{1}{2}\mathbf{v}_0^{\mathrm{T}}\frac{\partial^2 \mathbf{D}}{\partial k_r \partial k_s}\mathbf{u}_0, \quad r, s = 1, 2, \dots, n-1, \tag{22}$$

and the real matrix **G** is determined by the expression

$$\langle \mathbf{G}\mathbf{k}', \mathbf{k}'\rangle = \sum_{r=1}^{n-1} k'_r\mathbf{v}_0^{\mathrm{T}}\frac{\partial \mathbf{D}}{\partial k_r}\mathbf{S}_0\left(\sum_{s=1}^{n-1} k'_s\frac{\partial \mathbf{D}}{\partial k_s}\mathbf{u}_0\right), \tag{23}$$

where $\mathbf{S}_0$ is the operator inverse to the operator $\mathbf{A}(q_0) - \omega_0^2\mathbf{M}$.

In view of Eqs. (18) and (19), which explicitly specify the curve $\mathbf{p}(\varepsilon)$, and expansion (20), Eq. (21) is represented in the form

$$\begin{gathered}\Delta\lambda^2 - \Delta\lambda\omega_0\langle \mathbf{h}, \mathbf{k}\rangle + \tilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle \\ + i\omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle) = 0,\end{gathered} \tag{24}$$

where $\Delta\lambda = \lambda - i\omega_0$. According to the Bilharz criterion [10], all roots of polynomial (24) with complex coefficients have negative real parts iff

$$\tilde{f}(q - q_0) > \frac{(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle)^2}{\langle \mathbf{h}, \mathbf{k}\rangle^2} - \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle, \tag{25}$$

$$\langle \mathbf{h}, \mathbf{k}\rangle < 0. \tag{26}$$

Without loss of generality, we take $\tilde{f} < 0$. Then, the critical parameter q above which instability (flutter) occurs is given by the expression

$$q_{\mathrm{cr}}(\mathbf{k}) = q_0 + \frac{(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle)^2}{\tilde{f}\langle \mathbf{h}, \mathbf{k}\rangle^2} - \frac{\omega_0^2}{\tilde{f}}\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle. \tag{27}$$

Considering the case where

$$\{\mathbf{k}\colon q < q_{\rm cr}(\mathbf{k})\} \subset \{\mathbf{k}\colon \langle \mathbf{h}, \mathbf{k}\rangle < 0,\ \langle \mathbf{g}_s, \mathbf{k}\rangle > 0,\ s = 1, 2, \ldots, m-2\}, \tag{28}$$

i.e., all simple eigenvalues $\pm i\omega_{0,s}$ are displaced to the left-hand side of the complex plane, we conclude that the surface $q_{\rm cr}(\mathbf{k})$ approximated by Eq. (27) is the boundary of the asymptotic-stability region. According to Eq. (27), it is easy to see that the limit of $q_{\rm cr}$ as a function of $k_1, k_2, \ldots, k_{n-1}$ does not exist at the point $\mathbf{k} = 0$ due to singularity. This conclusion was first drawn in [6, 7], where the stability of a gyroscope in a gimbal and Ziegler pendulum was analyzed.

Setting $\mathbf{k} = \varepsilon\tilde{\mathbf{k}}$ in Eq. (27), we find the jump of the critical load as a function of $\tilde{\mathbf{k}}$:

$$\Delta q \equiv q_0 - \lim_{\varepsilon\to 0} q_{\rm cr}(\varepsilon\tilde{\mathbf{k}}) = -\frac{1}{\tilde{f}}\frac{\langle \mathbf{f}, \tilde{\mathbf{k}}\rangle^2}{\langle \mathbf{f}, \tilde{\mathbf{k}}\rangle^2}* \tag{29}$$

The limit of $q_{\rm cr}$ along the $\tilde{\mathbf{k}}$ direction exists if $\langle \mathbf{h}, \tilde{\mathbf{k}}\rangle \neq 0$, because the numerator and denominator in Eq. (29) are homogeneous. When $\langle \mathbf{f}, \tilde{\mathbf{k}}\rangle = 0$, the jump of the critical load vanishes. This condition provides the ratio $\frac{k_i}{k_j} = -\frac{f_j}{f_i}$, where $i, j = 1$ and 2, of the components k_1 and k_2 of the two-dimensional vector $\mathbf{k} = (k_1, k_2)$ for which the circulatory system is not destabilized by weak dissipative and gyroscopic forces. The dependence of the critical load on the ratio of the dissipation parameters was first found by Bolotin [2, 3]. For two-dimensional system (1) with the matrix $\mathbf{D}(\mathbf{k}) = k\tilde{\mathbf{D}}$, where $\tilde{\mathbf{D}}$ is a fixed matrix, expression (29) describing the jump of the critical load takes the form

$$\Delta q = \frac{-2({\rm tr}\mathbf{A}_0)^2}{2{\rm tr}\mathbf{A}_0\mathbf{A}_1 - {\rm tr}\mathbf{A}_0{\rm tr}\mathbf{A}_1}\left(\frac{2{\rm tr}\mathbf{A}_0\tilde{\mathbf{D}} - {\rm tr}\mathbf{A}_0{\rm tr}\tilde{\mathbf{D}}}{2{\rm tr}\mathbf{A}_0\tilde{\mathbf{D}} - 3{\rm tr}\mathbf{A}_0{\rm tr}\tilde{\mathbf{D}}}\right)^2,$$

where $\mathbf{A}_0 = \mathbf{A}(q_0)$, $\mathbf{A}_1 = \frac{d\mathbf{A}}{dq}$, and the derivative is taken at $q = q_0$.

The isolines of function (27) are the boundaries of the asymptotic-stability region in the space of the parameters $\mathbf{k} = (k_1, k_2, \ldots, k_{n-1})$. The isolines $q_{\rm cr} = q_0$, where q_0 is the critical parameter q for the unperturbed circulatory system, are given by the expression

$$\langle \mathbf{f}, \mathbf{k}\rangle = \pm\omega_0\langle \mathbf{h}, \mathbf{k}\rangle\sqrt{\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle} - \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle. \tag{30}$$

For the two-dimensional parameter vector $\mathbf{k} = (k_1, k_2)$, Eq. (27) describes a surface known as the Whitney umbrella [9]. Expressing the parameter k_1 in terms of k_2 from Eq. (30) and vice versa, we obtain an approximation of the boundary of the asymptotic-stability region on the (k_1, k_2) plane for $q_{\rm cr} = q_0$ in the form

$$k_i = -\frac{f_j}{f_i}k_j - \frac{\mathbf{f}^{\rm T}\mathbf{H}^*\mathbf{f} \pm \omega_0(h_i f_j - h_j f_i)\sqrt{\mathbf{f}^{\rm T}\mathbf{G}^*\mathbf{f}}}{f_i^3}k_j^2 + o(k_j^2), \quad i, j = 1, 2, \tag{31}$$

$$\mathbf{H}^* = \begin{bmatrix} H_{22} & -H_{12} \\ -H_{21} & H_{11}\end{bmatrix}, \quad \mathbf{G}^* = \begin{bmatrix} G_{22} & -G_{12} \\ -G_{21} & G_{11}\end{bmatrix},$$

where the matrices $\mathbf{H}$ and $\mathbf{G}$ with the respective components H_{rs} and G_{rs}, where $r, s = 1$ and 2, are given by expressions (22) and (23), respectively. It follows from Eq. (31) that the asymptotic-stability region has a singularity—turning point—at the origin. The asymptotic expression for the stabilization-region boundary in form (31) was first found in [7] for the Ziegler pendulum with two independent dissipation parameters.

Substituting $\Delta\lambda = {\rm Re}\lambda + i({\rm Im}\lambda - \omega_0)$ into Eq. (24) and separating the real and imaginary parts, we arrive at the following equations describing the displacement of the eigenvalues λ on the complex plane under small variations of the parameters q and $\mathbf{k}$:

$$\left({\rm Im}\,\lambda - \omega_0 - {\rm Re}\lambda - \frac{a}{2}\right)^2 - \left({\rm Im}\,\lambda - \omega_0 + {\rm Re}\lambda + \frac{a}{2}\right)^2 = 2d, \tag{32}$$

$$\left({\rm Re}\lambda + \frac{a}{2}\right)^4 + \left(c - \frac{a^2}{4}\right)\left({\rm Re}\lambda + \frac{a}{2}\right)^2 = \frac{d^2}{4}, \tag{33}$$

$$({\rm Im}\,\lambda - \omega_0)^4 - \left(c - \frac{a^2}{4}\right)({\rm Im}\,\lambda - \omega_0)^2 = \frac{d^2}{4}, \tag{34}$$

where $a = -\omega_0\langle \mathbf{h}, \mathbf{k}\rangle$, $c = \tilde{f}(q - q_0) + \omega_0^2\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle$, and $d = \omega_0(\langle \mathbf{f}, \mathbf{k}\rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle)$. According to Eqs. (32)–(34) for $\mathbf{k} = 0$, when varying the parameter q, two simple imaginary eigenvalues merge at $q = q_0$ and then split in the direction perpendicular to the imaginary axis with the formation of a pair of simple complex eigenvalues (flutter). Such a behavior of the eigenvalues is known as *strong interaction* and is typical for the circulatory system [5]. The trajectories of the eigenvalues of the circulatory system when varying the parameter q are shown in Fig. 1 by the thin lines (for $\tilde{f} < 0$).

For $\mathbf{k} \neq 0$ and $d \neq 0$, dissipative and gyroscopic forces destroy strong interaction by displacing and splitting the trajectories of the eigenvalues of the circulatory system, as is shown in Fig. 1a. This effect, which was previously known only qualitatively [2–4], is analytically described by Eqs. (32)–(34). When varying the parameter q at a fixed vector $\mathbf{k}$, the eigenvalues are displaced along the branches of hyperbola (32) on the complex plane. This hyperbola has two asymptotes, $\operatorname{Re}\lambda = -\frac{a}{2}$ and $\operatorname{Im}\lambda = \omega_0$. If $a > 0$, one of the two eigenvalues lies on the left-hand side of the complex plane, and the second eigenvalue is displaced to the right-hand side at the critical load q_{cr} determined by Eq. (27).

For $d = 0$, the strong interaction between eigenvalues holds despite the introduction of weak velocity-dependent forces ($\mathbf{k} \neq 0$). According to Eq. (33), the complex eigenvalues with $\operatorname{Re}\lambda = -\frac{a}{2}$ are strongly coupled with each other for q equal to

$$q_* = q_0 + \omega_0^2 \frac{\langle \mathbf{h}, \mathbf{k} \rangle^2 - 4\langle \mathbf{Gk}, \mathbf{k} \rangle}{4\tilde{f}}. \qquad (35)$$

When varying the parameter q, the double eigenvalue $\lambda_* = -\frac{a}{2} + i\omega_0$ splits into two simple complex-conjugated eigenvalues, one of which intersects the imaginary axis at the critical value given by Eq. (27), as is shown in Fig. 1b for $a > 0$. In this case, weak forces dependent on the generalized velocities stabilize the circulatory system when $\langle \mathbf{Gk}, \mathbf{k} \rangle > 0$.

4. Let us consider the Reut–Sugiyama pendulum [11] consisting of two rigid rods that have the same length l and equal unit-length mass m. The rods are joined together by a hinge. A plane rigid massless plate is fixed in the free end of one rod perpendicularly to it. The pendulum is subject to the force Q always directed along the vertical axis that is the equilibrium position of the pendulum (Fig. 2). This system was realized under laboratory conditions, and the force Q was generated by the pressure of an air jet [11]. Viscoelastic hinges of the pendulum are characterized by the same rigidity c and viscosity b. The external damping coefficient due to air resistance is denoted as e. In the dimensionless quantities

$$q = \frac{Ql}{c}, \quad \gamma = \frac{b}{l\sqrt{cml}}, \quad \kappa = \frac{el^2}{\sqrt{cml}}, \quad \tau = \frac{t}{l}\sqrt{\frac{c}{ml}}, \qquad (36)$$

the equation of small oscillations of the pendulum has the form [11]

$$\ddot{\mathbf{y}} + \mathbf{D}\dot{\mathbf{y}} + \mathbf{A}\mathbf{y} = 0, \qquad (37)$$

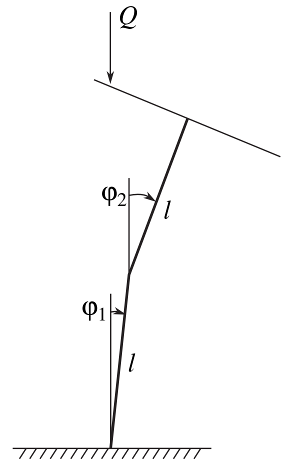

Fig. 2. Reut–Sugiyama pendulum.

where

$$\mathbf{D} = \frac{1}{7}\begin{bmatrix} 7\kappa + 42\gamma & -30\gamma \\ -84\gamma & 7\kappa + 66\gamma \end{bmatrix},$$
$$\mathbf{A} = \frac{1}{7}\begin{bmatrix} 42 - 30q & -30 \\ 66q - 84 & 66 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}. \qquad (38)$$

In the absence of dissipation ($\gamma = \kappa = 0$), the equilibrium position of the pendulum is stable if $0 \le q < q_0$. Under the load $q_0 = \frac{18 - 2\sqrt{7}}{5} \simeq 2.54170$ [11], there are a pair of double eigenvalues $\pm i\omega_0$, where $\omega_0 = 6^{1/2}7^{-1/4}$, corresponding to one eigenvector.

The Routh–Hurwitz conditions applied to the system of Eqs. (37) and (38) provides the equation for the boundary of the asymptotic-stability region for $q_{cr} = q_0$ in the form

$$(36\gamma^2 + 108\gamma\kappa + 7\kappa^2)(54\gamma + 7\kappa)(6\gamma + \kappa\sqrt{7})$$
$$= 216(9 - \sqrt{7})^2\gamma^2, \quad \kappa > -\frac{54}{7}\gamma, \quad \kappa > -\frac{6\sqrt{7}}{7}\gamma. \qquad (39)$$

The asymptotic-stability region with the boundary given by Eq. (39) is shaded in Fig. 3. It approaches the origin as a narrow tongue along the vertical axis, which illustrates the stabilizing and destabilizing effects of low external κ and internal γ dampings, respectively [3, 4]. Indeed, Fig. 3 shows that, for any infinitely small γ value, there is a κ value such that the perturbed nonconservative system is stable with $q_{cr}(\gamma, \kappa) > q_0$.

Let us use the above results to approximate the asymptotic-stability region near the origin as well as to describe the behavior of the eigenvalues. Solving prob-

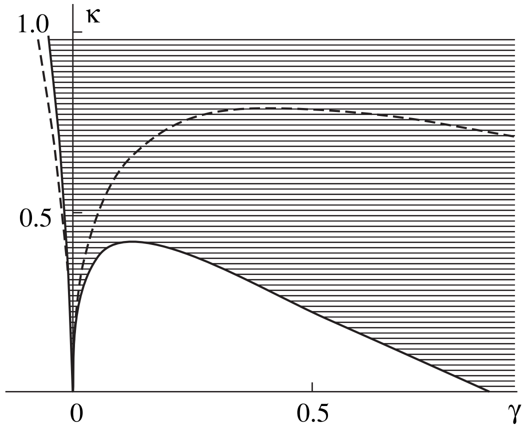

Fig. 3. Asymptotic-stability region is shaded on the plane of the parameters of internal γ and external κ damping for $q_{cr} = q_0$.

lems given by Eqs. (5) and (6) for $q_0 = \dfrac{18 - 2\sqrt{7}}{5}$ and $\omega_0 = 6^{1/2}7^{-1/4}$ with the matrices **D** and **A** specified by Eqs. (38), we obtain the Jordan vector chains for the double eigenvalue $i\omega_0$:

$$\mathbf{u}_0 = \frac{\sqrt{7}}{140}\begin{bmatrix} -5 \\ 11 - \sqrt{7} \end{bmatrix}, \quad \mathbf{v}_0 = \begin{bmatrix} 11 - \sqrt{7} \\ 5 \end{bmatrix},$$
$$\mathbf{u}_1 = \frac{i\omega_0}{120}\begin{bmatrix} -5 \\ 11 - 3\sqrt{7} \end{bmatrix}, \quad \mathbf{v}_1 = \frac{7i\omega_0}{3}\begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{40}$$

The substitution of these vectors into expressions (11), (12), (22), and (23) yields

$$\mathbf{f} = \frac{3}{7}\begin{bmatrix} 9 - \sqrt{7} \\ 0 \end{bmatrix}, \quad \tilde{f} = -\frac{15}{14}, \quad \mathbf{h} = -\frac{\omega_0\sqrt{7}}{42}\begin{bmatrix} 3(9 + \sqrt{7}) \\ 7 \end{bmatrix},$$
$$\mathbf{H} \equiv 0, \quad \mathbf{G} = \frac{1}{24}\begin{bmatrix} 0 & 3 \\ 3 & \sqrt{7} \end{bmatrix}. \tag{41}$$

In view of expressions (41), expression (27) for the critical load takes the form

$$q_{cr}(\gamma, \kappa) = q_0 - \frac{\sqrt{7}}{30}\frac{216(9 - \sqrt{7})^2\gamma^2}{((9 + \sqrt{7})3\gamma + 7\kappa)^2} + \frac{\sqrt{7}}{30}(6\gamma\kappa + \sqrt{7}\kappa^2). \tag{42}$$

Approximation of the stability-region boundary given by Eq. (39) that follows from Eq. (42) for $q_{cr} = q_0$ is shown by the dashed line in Fig. 3. According to Eq. (42), the critical load for $\gamma = 0$ increases with external damping as $q_{cr} = q_0 + \dfrac{7\kappa^2}{30}$, which agrees with the shape of the stability region shown in Fig. 3.

The substitution of Eq. (41) into Eqs. (32) and (33) provides explicit expressions describing the trajectories of the eigenvalues on the complex plane as well as the behavior of their real parts when varying the load parameter q:

$$\left(\mathrm{Im}\,\lambda - \omega_0 + \mathrm{Re}\,\lambda + \frac{3(9 + \sqrt{7})\gamma + 7\kappa}{14}\right)^2 - \left(\mathrm{Im}\,\lambda - \omega_0 - \mathrm{Re}\,\lambda - \frac{3(9 + \sqrt{7})\gamma + 7\kappa}{14}\right)^2 = \gamma\frac{6(\sqrt{7} - 9)\omega_0}{7}, \tag{43}$$

$$\left(\mathrm{Re}\,\lambda + \frac{3(9 + \sqrt{7})\gamma + 7\kappa}{14}\right)^4 - \frac{3}{14}\left(5(q - q_0) + \frac{3\gamma}{7}((44 + 9\sqrt{7})\gamma + 21\kappa)\right) \times \left(\mathrm{Re}\,\lambda + \frac{3(9 + \sqrt{7})\gamma + 7\kappa}{14}\right)^2 = \gamma^2\frac{27(44\sqrt{7} - 63)}{343}. \tag{44}$$

For $\kappa = \gamma = 0$, the eigenvalues are strongly coupled at $q = q_0$. In the absence of internal damping ($\gamma = 0$), the double eigenvalue is displaced to the left-hand side of the complex plane due to external damping κ stabilizing the circulatory system. In this case, $\tilde{q}_{cr} = q_0$ according to Eqs. (29) and (42). In the absence of external damping ($\kappa = 0$), internal damping γ destroys strong interaction and displaces frequency curves to the left-hand side of the complex plane. In this case, the Reut–Sugiyama pendulum is destabilized by internal damping, and, according to Eqs. (29) and (42), the jump of the critical load for $\gamma \longrightarrow 0$ is equal to

$$\Delta q = \frac{4\sqrt{7}}{5}\frac{44 - 9\sqrt{7}}{44 + 9\sqrt{7}} \simeq 0.63013, \tag{45}$$

which agrees satisfactorily with the exact value $\Delta q = \dfrac{2(44 - 9\sqrt{7})}{45} \simeq 0.89725$ [11]. The approximate jump value $\Delta q = 0.43733$ for $\dfrac{\gamma}{\kappa} = 1$ and $\gamma \longrightarrow 0$ is quite close to the exact value $\Delta q = 0.42198$. Thus, the approximation of the critical-load jump by Eqs. (29) and (42) is improved with decreasing the ratio $\dfrac{\gamma}{\kappa}$ to zero.

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