Gyroscopic stabilization in presence of non-conservative forces

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Abstract— *Stability of a linear autonomous non-conservative system in presence of potential, gyroscopic, dissipative, and non-conservative positional forces is studied. It is known that marginal stability of a gyroscopic system can be destroyed or improved up to asymptotic stability due to action of small non-conservative positional and damping forces. The present paper shows that the boundary of the asymptotic stability domain of the perturbed system possesses singularities such as "Dihedral angle" and "Whitney umbrella" that govern stabilization and destabilization. Approximations of the stability boundary near the singularities and estimates of the critical gyroscopic parameter are found in an analytic form. As an example, stability of the Crandall gyropendulum with stationary and rotating damping is considered in detail.*

Keywords: gyroscopic stabilization, non-conservative perturbation, multiple eigenvalue, stability boundary, singularity

I. Introduction

Consider an autonomous non-conservative system described by a linear differential equation of second order

$$\ddot{\mathbf{x}} + (\gamma\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + (\nu\mathbf{N} + \mathbf{P})\mathbf{x} = 0, \tag{1}$$

where dot denotes time differentiation, $\mathbf{x} \in R^m$, and real matrices $\mathbf{D} = \mathbf{D}^T$, $\mathbf{G} = -\mathbf{G}^T$, and $\mathbf{N} = -\mathbf{N}^T$ are related to dissipative (damping), gyroscopic, and non-conservative positional (circulatory) forces with magnitudes controlled by scaling factors δ, γ, and ν respectively, and real matrix $\mathbf{P} = \mathbf{P}^T$ corresponds to potential forces [1–33].

For $\mathbf{P} = -\mathbf{K} < 0$, the trivial solution to equation (1) is statically unstable (divergence) in the absence of dissipative, gyroscopic, and circulatory forces. A phenomenon of *gyroscopic stabilization* is that for even m, a statically unstable potential system can be made stable in the sense of Lyapunov by gyroscopic forces only ($\delta = \nu = 0$), if $\det \mathbf{G} \neq 0$ and the absolute value of the gyroscopic parameter is sufficiently large ($|\gamma| > \gamma_0$), see e.g. [1], [11], [14], [17], [18], [20], [21].

It is known that for $m = 2$ and $\det \mathbf{G} = 1$ the critical value of the gyroscopic parameter is simply [18], [32]

$$\gamma_0 = \sqrt{\mathrm{Tr}\mathbf{K} + 2\sqrt{\det \mathbf{K}}} = \sqrt{\lambda_1(\mathbf{K})} + \sqrt{\lambda_2(\mathbf{K})} > 0, \tag{2}$$

where $\lambda_1(\mathbf{K})$ and $\lambda_2(\mathbf{K})$ are eigenvalues of the real symmetric matrix $\mathbf{K} > 0$.

The problem of estimation of γ_0 for the case $m > 2$ is much more complicated, and in general it should be solved numerically, see [11], [14], [18], [20], [21] and references therein.

The marginal stability of a potential gyroscopic system is destroyed due to action of the dissipative and non-conservative positional forces. The latter, known also as circulatory forces, typically appear in systems with follower loads caused, e.g., by the jet thrust [6], [7], [8], [9], [14], [19] or by friction in contact [2], [3], [4], [17], [24], [25], [26], [28], [33]. Instead, the dissipative and circulatory forces can make the gyroscopic system asymptotically stable. The critical value of the gyroscopic parameter at the onset of the gyroscopic stabilization $\gamma_{cr}(\delta, \nu)$ is then a function of parameters, corresponding to the non-conservative forces, and can differ dramatically from γ_0 [17], [32].

In spite of the fact that the effect of dissipative and non-conservative positional forces on stability of rotors, which are statically stable when non rotate, has a long history, see e.g. [2], [3], [4], [5], [7], [9], [10], [14], [16], the problem of gyroscopic stabilization in presence of such forces seems to attract considerable attention of researchers only in recent years [17], [25], [26], [28], [32], [33].

The goal of the present paper is to study the effect of weak damping and non-conservative positional forces on the onset of the gyroscopic stabilization and to obtain the estimates of the critical gyroscopic parameter $\gamma_{cr}(\delta, \nu)$.

II. Gyroscopic stabilization of a potential system

In our subsequent considerations we assume that in the absence of dissipative and non-conservative positional forces ($\delta = \nu = 0$) the gyroscopic stabilization of the system (1) occurs for $\gamma > \gamma_0 > 0$. This means that all the eigenvalues λ of the eigenvalue problem

$$(\mathbf{I}\lambda^2 + \lambda\gamma\mathbf{G} - \mathbf{K})\mathbf{u} = 0, \tag{3}$$

where $\mathbf{I}$ is the identity matrix, are purely imaginary and semi-simple. At $\gamma = \gamma_0$ there exists a pair of double purely imaginary eigenvalues $\lambda = \pm i\omega_0$ with the Jordan chain of length 2, other eigenvalues being simple and purely imaginary [17], [18], [20], [21].

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The Jordan chain at the eigenvalue $i\omega_0$ consists of the eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$, which solve the following equations [18], [29]

$$(-\mathbf{I}\omega_0^2 + i\omega_0\gamma_0\mathbf{G} - \mathbf{K})\mathbf{u}_0 = 0, \quad (4)$$

$$(-\mathbf{I}\omega_0^2 + i\omega_0\gamma_0\mathbf{G} - \mathbf{K})\mathbf{u}_1 = -(2i\omega_0\mathbf{I} + \gamma_0\mathbf{G})\mathbf{u}_0. \quad (5)$$

The eigenvector $\mathbf{u}_0$ satisfies the orthogonality condition

$$\mathbf{u}_0^*(2i\omega_0\mathbf{I} + \gamma_0\mathbf{G})\mathbf{u}_0 = 0, \quad (6)$$

where the asterisk denotes Hermitian conjugate. With the help of equation (4) the orthogonality condition (6) yields the Rayleigh quotient for the critical frequency

$$\omega_0^2 = \frac{\mathbf{u}_0^*\mathbf{K}\mathbf{u}_0}{\mathbf{u}_0^*\mathbf{u}_0}. \quad (7)$$

In the vicinity of $\gamma = \gamma_0$, the double eigenvalue and the corresponding eigenvector are changing according to the formulae

$$i\omega(\gamma) = i\omega_0 \pm i\mu\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0), \quad (8)$$

$$\mathbf{u}(\gamma) = \mathbf{u}_0 \pm i\mu\mathbf{u}_1\sqrt{\gamma - \gamma_0} + O(\gamma - \gamma_0), \quad (9)$$

where μ^2 is a real quantity

$$\mu^2 = \frac{i\omega_0\mathbf{u}_0^*\mathbf{G}\mathbf{u}_0}{2i\omega_0\mathbf{u}_0^*\mathbf{u}_1 + \gamma_0\mathbf{u}_0^*\mathbf{G}\mathbf{u}_1 + \mathbf{u}_0^*\mathbf{u}_0} \quad (10)$$

$$= -\frac{2\omega_0^2\mathbf{u}_0^*\mathbf{u}_0}{\gamma_0(\omega_0^2\mathbf{u}_1^*\mathbf{u}_1 + \mathbf{u}_1^*\mathbf{K}\mathbf{u}_1 - i\omega_0\gamma_0\mathbf{u}_1^*\mathbf{G}\mathbf{u}_1 - \mathbf{u}_0^*\mathbf{u}_0)}, \quad (11)$$

see e.g. [29].

For $m = 2$ and $\det \mathbf{G} = 1$, at the critical value of the gyroscopic parameter γ_0 defined by equation (2), the double eigenvalue

$$i\omega_0 = i\sqrt[4]{\det \mathbf{K}} \quad (12)$$

has the Jordan chain consisting of the eigenvector $\mathbf{u}_0$ and associated vector $\mathbf{u}_1$

$$\mathbf{u}_0 = C\begin{pmatrix} -i\omega_0\gamma_0 + k_{12} \\ -\omega_0^2 - k_{11} \end{pmatrix}, \quad (13)$$

$$\mathbf{u}_1 = -\frac{C}{\omega_0^2 + k_{22}}\begin{pmatrix} 0 \\ i\omega_0(k_{11} - k_{22}) + \gamma_0 k_{12} \end{pmatrix}, \quad (14)$$

where C is a complex coefficient. The vector $\mathbf{u}_1$ is defined up to an addend proportional to $\mathbf{u}_0$. Substituting the vectors (13) and (14) into equation (10) and taking into account the orthogonality condition (6), we get

$$\mu^2 = \frac{\gamma_0}{2}\frac{(k_{11} + \omega_0^2)(k_{22} + \omega_0^2)}{\gamma_0^2\omega_0^2 + k_{12}^2} = \frac{\gamma_0}{2} > 0. \quad (15)$$

Therefore, the coefficient $\mu = \sqrt{\gamma_0/2}$ is a real quantity and according to the formula (8), for $\gamma > \gamma_0$ the double eigenvalue splits into two simple purely imaginary eigenvalues (gyroscopic stabilization). As it follows from expressions (8) and (11), the same mechanism governs the gyroscopic stabilization of a statically unstable potential system in the case of arbitrary even m, see e.g. [18], [21].

III. Gyroscopic stabilization of a non-potential system

The most interesting for many applications is the situation when system (1) is weakly non-potential with $\delta \sim \nu \ll \gamma \sim \gamma_0$. Furthermore, the effect of small damping and non-conservative positional forces on the stability of gyroscopic systems is regarded as *paradoxical*, since the stability properties are extremely sensitive to the choice of the perturbation, and the balance of forces resulting in the asymptotic stability is not evident [7], [17], [33]. This characterization sounds even more justified if to take into account the connection of the *destabilization paradox* with the physical paradoxes such as "tippe top inversion" and "rising egg phenomenon" [25], [26], [28], [33].

Perturbing the system (1), which is stabilized by the gyroscopic forces with $\gamma > \gamma_0$, by small damping and circulatory forces, yields an increment to a purely imaginary eigenvalue $i\omega(\gamma)$ [13], [16], [29], [32]

$$\lambda = i\omega - \frac{i\omega\mathbf{u}^*\mathbf{D}\mathbf{u}\delta + \mathbf{u}^*\mathbf{N}\mathbf{u}\nu}{2i\omega\mathbf{u}^*\mathbf{u} + \mathbf{u}^*\mathbf{G}\mathbf{u}\gamma} + o(\delta, \nu) \quad (16)$$

$$= i\omega + \frac{\omega^2\mathbf{u}^*\mathbf{D}\mathbf{u}\delta - i\omega\mathbf{u}^*\mathbf{N}\mathbf{u}\nu}{\mathbf{u}^*\mathbf{K}\mathbf{u} - \omega^2\mathbf{u}^*\mathbf{u}} + o(\delta, \nu). \quad (17)$$

where $\mathbf{u}(\gamma)$ is the eigenvector corresponding to $i\omega(\gamma)$.

Since $\mathbf{D}$ and $\mathbf{K}$ are real symmetric matrices and $\mathbf{N}$ is a real skew-symmetric one, the increment to the eigenvalue $i\omega(\gamma)$ due to action of small damping and circulatory forces is a real quantity. When it is negative, the purely imaginary eigenvalue moves to the left side of the complex plane due to action of small damping and circulatory forces. If at a given γ all the purely imaginary eigenvalues take negative increments, the system (1) is asymptotically stable.

Consequently, in the first approximation with respect to δ and ν, a simple purely imaginary eigenvalue $i\omega(\gamma)$ remains on the imaginary axis, if

$$\nu = \beta(\gamma)\delta, \quad (18)$$

where

$$\beta(\gamma) = -i\omega(\gamma)\frac{\mathbf{u}^*(\gamma)\mathbf{D}\mathbf{u}(\gamma)}{\mathbf{u}^*(\gamma)\mathbf{N}\mathbf{u}(\gamma)}. \quad (19)$$

In general, for a given γ, two lines of the kind (18), corresponding to two different purely imaginary eigenvalues, form the linear approximation to the boundary of the asymptotic stability domain of the weakly non-potential gyroscopic system (1) near the origin in the plane of the parameters δ and ν.

If the functions $\omega(\gamma)$ and $\mathbf{u}(\gamma)$ are known in expression (19), then equation (18) gives the linear approximation to the asymptotic stability domain in the space of three parameters δ, ν, and γ. In [32] the functions $\omega(\gamma)$ and $\mathbf{u}(\gamma)$ of the unperturbed gyroscopic system with $m = 2$ degrees of freedom were found exactly and used for the calculation of the stability boundary and the critical value of the gyroscopic parameter $\gamma_{cr}(\delta, \nu)$. In the earlier work [17] an analogous approach has been used for the study of stability of a gyroscopic pendulum.

We note that in general explicit analytical expressions for $\omega(\gamma)$ and $\mathbf{u}(\gamma)$ cannot be derived. Instead, numerical data can be used as it has been done in [8]. An alternative way is to use approximate analytical expressions for eigenfrequencies and eigenfunctions, obtained by the methods of perturbation theory.

In the vicinity of $\gamma = \gamma_0$ all simple eigenvalues $i\omega(\gamma)$ vary slowly when γ is changing, except for the two eigenvalues that coincide at $\gamma = \gamma_0$, as it follows from expansions (8). It is intuitively clear that the behavior of these two eigenvalues due to small non-conservative perturbation is defining for the stability of a weakly non-potential gyroscopic system. For this reason, we estimate the critical value of the gyroscopic parameter at the onset of gyroscopic stabilization of system (1), substituting the expansions (8) and (9) in expression (19)

$$\beta(\gamma) = -(i\omega_0 \pm i\mu\sqrt{\gamma-\gamma_0})\frac{\mathbf{u}^*(\gamma)\mathbf{D}\mathbf{u}(\gamma)}{\mathbf{u}^*(\gamma)\mathbf{N}\mathbf{u}(\gamma)}$$

$$= -(\omega_0 \pm \mu\sqrt{\gamma-\gamma_0})\frac{d_1 \mp \mu d_2\sqrt{\gamma-\gamma_0}}{n_1 \pm \mu n_2\sqrt{\gamma-\gamma_0}}, \qquad (20)$$

where real scalars d_1, d_2, n_1, and n_2 are

$$d_1 = \mathrm{Re}(\mathbf{u}_0^*\mathbf{D}\mathbf{u}_0), \quad d_2 = \mathrm{Im}(\mathbf{u}_0^*\mathbf{D}\mathbf{u}_1 - \mathbf{u}_1^*\mathbf{D}\mathbf{u}_0), \quad (21)$$

$$n_1 = \mathrm{Im}(\mathbf{u}_0^*\mathbf{N}\mathbf{u}_0), \quad n_2 = \mathrm{Re}(\mathbf{u}_0^*\mathbf{N}\mathbf{u}_1 - \mathbf{u}_1^*\mathbf{N}\mathbf{u}_0). \quad (22)$$

From formula (20) it follows that the new critical value is given by the expression

$$\gamma_{cr}(\beta) = \gamma_0 + \frac{n_1^2(\beta-\beta_0)^2}{\mu^2(\omega_0 d_2 - \beta_0 n_2 - d_1)^2}, \qquad (23)$$

which is valid for $\beta - \beta_0 \ll 1$, where β_0 is the value of the function $\beta(\gamma)$ at the onset of the gyroscopic stabilization for the potential gyroscopic system

$$\beta_0 = \beta(\gamma_0) = -i\omega_0\frac{\mathbf{u}_0^*\mathbf{D}\mathbf{u}_0}{\mathbf{u}_0^*\mathbf{N}\mathbf{u}_0}. \qquad (24)$$

Substituting $\beta = \nu/\delta$ in expression (23) yields the equation

$$\gamma_{cr}(\delta,\nu) = \gamma_0 + \frac{n_1^2(\nu-\beta_0\delta)^2}{\mu^2(\omega_0 d_2 - \beta_0 n_2 - d_1)^2\delta^2} \geq \gamma_0, \quad (25)$$

which together with formula (24) constitutes the central result of our work. The equations obtained give a simple approximation of the boundary of the asymptotic stability domain of the weakly non-potential gyroscopic system in the vicinity of the point $(0, 0, \pm\gamma_0)$ in the space of the parameters (δ, ν, γ), as well as provide an estimate of the critical value of the gyroscopic parameter $\gamma_{cr}(\delta, \nu)$.

It is remarkable that equation (25) has the form $Z = X^2/Y^2$, which is canonical for the singular surface known as the Whitney umbrella [22], [32], [34]. This fact supports the qualitative picture established by Arnold [34], that the boundary of the asymptotic stability domain of a multiparameter family of real matrices is not a smooth surface. Generically it possesses singularities corresponding to multiple eigenvalues with zero real part. In particular, for real matrices depending on three parameters, two different pairs of simple purely imaginary eigenvalues originate a singularity of the stability boundary, which is shaped as a *dihedral angle* in the parameter space Fig. 1(a). A pair of double purely imaginary eigenvalues with the Jordan block corresponds to the singularity *deadlock of an edge*, which is a half of the *Whitney umbrella* surface [34], see Fig. 1(b).

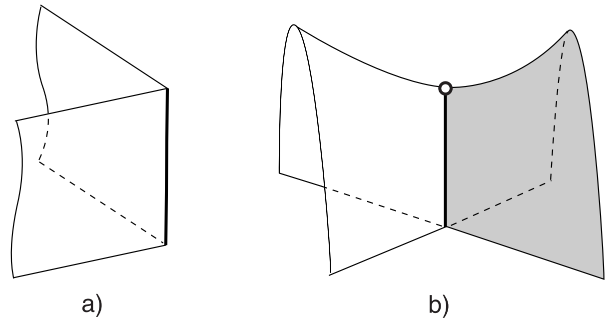

Fig. 1. The singularities ”dihedral edge” (a) and ”Whitney umbrella”.

Expression (25) explicitly shows that the function $\gamma_{cr}(\delta, \nu)$ is non-differentiable at the origin and depends only on the ratio $\beta = \nu/\delta$. Therefore, the limit of $\gamma_{cr}(\delta, \nu)$ at the origin is not defined and strongly depends on the direction of approaching given by β. Most of the directions β give the limit value $\gamma_{cr}(\beta) > \gamma_0$. The latter means that the critical ”angular velocity” γ generally jumps up for infinitely small δ and ν. Such ”jumps” illustrating the high sensitivity of the critical parameters responsible for the onset of the flutter instability to small imperfections are caused by the Whitney umbrella singularity of the domain of the asymptotic stability of a non-conservative gyroscopic system.

We note that the formulae (24) and (25) generalize the result of [32], obtained for the non-potential gyroscopic system (1) with $m = 2$ and $\det \mathbf{G} = 1$, to the case of arbitrary even m and $\det \mathbf{G} \neq 0$. Indeed, for $m = 2$ and $\det \mathbf{G} = 1$, computation of β_0 by formula (24) with the use of the eigenvector (13) yields

$$\beta_0 = \frac{d_{11}(\omega_0^2 + k_{22}) - 2d_{12}k_{12} + d_{22}(\omega_0^2 + k_{11})}{2\gamma_0}$$

$$= \frac{\mathrm{Tr}\left[(\gamma_0^2 - \omega_0^2)\mathbf{I} - \mathbf{K}\right]\mathbf{D}}{2\gamma_0}, \qquad (26)$$

in agreement with [32]. Substituting the eigenvector (13) and associated eigenvector (14) into expressions (21) and

(22), we find

$$n_1 = -2\gamma_0\omega_0(\omega_0^2 + k_{11}), \tag{27}$$

$$n_2 = -2\gamma_0\frac{\omega_0^2(k_{22}-k_{11}) + k_{12}^2}{\omega_0^2 + k_{22}}, \tag{28}$$

$$d_1 = 2\gamma_0\beta_0(\omega_0^2 + k_{11}), \tag{29}$$

$$\frac{d_2}{2\omega_0} = \frac{(d_{12}k_{12} - d_{22}(\omega_0^2 + k_{11}))(k_{22}-k_{11}) - d_{12}k_{12}\gamma_0^2}{\omega_0^2 + k_{22}}. \tag{30}$$

Taking into account that $\gamma_0^2 = \mathrm{Tr}\mathbf{K} + 2\omega_0^2$, and using the relations (27)–(30) we calculate the expression in the denominator of formula (25)

$$\begin{aligned}
&\omega_0 d_2 - \beta_0 n_2 - d_1 = \\
&= 2\omega_0^2\frac{(k_{22}-k_{11}-\gamma_0^2)(\beta_0\gamma_0 + d_{12}k_{12})}{\omega_0^2 + k_{22}} \\
&-2\omega_0^2\frac{d_{22}(\omega_0^2 + k_{11})(k_{22}-k_{11})}{\omega_0^2 + k_{22}} \\
&= \omega_0^2\frac{(d_{11}(\omega_0^2 + k_{22}) - d_{22}(\omega_0^2 + k_{11}))(k_{22}-k_{11})}{\omega_0^2 + k_{22}} \\
&-\omega_0^2\frac{\gamma_0^2(d_{11}(\omega_0^2 + k_{22}) + d_{22}(\omega_0^2 + k_{11}))}{\omega_0^2 + k_{22}} \\
&= -2\omega_0^2(\omega_0^2 + k_{11})\mathrm{Tr}\mathbf{D}.
\end{aligned} \tag{31}$$

Then, substituting expressions (15), (27) and (31) into equation (25) we find that [32]

$$\gamma_{cr}(\nu,\delta) = \gamma_0 + \gamma_0\frac{2}{(\omega_0\mathrm{Tr}\mathbf{D})^2}\frac{(\nu - \beta_0\delta)^2}{\delta^2} \tag{32}$$

where β_0 is given by formula (26). We note that the equations (26) and (32) have been derived in [32] from the stability conditions of Routh and Hurwitz.

IV. Stability of a gyropendulum with stationary and rotating damping

As an example we consider the Crandall gyropendulum [17], [32]. The pendulum is an axisymmetric rigid body pivoted at a point O on the axis as shown in Fig. 2. When the axial spin Ω is absent, the upright position is statically unstable. When $\Omega \neq 0$ the body becomes a gyroscopic pendulum. Its primary parameters are its mass m, the distance L between the mass center and the pivot point, the axial moment of inertia I_a, and the diametral moment of inertia I_d about the pivot point. The gravity acceleration is denoted by g.

It is assumed that a drag force proportional to the linear velocity of the center of mass of the gyropendulum acts at the center of mass to oppose that velocity (stationary damping with the coefficient b_s). Additionally, it is assumed that

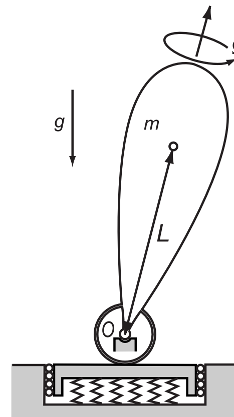

Fig. 2. The Crandall gyropendulum.

a rigid sphere concentric with the pendulum tip O, is attached to the pendulum and rubs against a fixed rub plate. The gyropendulum is supported frictionlessly at O, while a viscous friction force acts between the larger sphere and the rub plate, being responsible for the rotating damping with the coefficient b_r. The linearized equations of motion for the gyropendulum in the vicinity of the vertical equilibrium position derived in [17] have the form (1) with the matrices $\mathbf{G}$, $\mathbf{D}$, $\mathbf{K}$, and $\mathbf{N}$ given by the expressions

$$\gamma\mathbf{G} = \begin{pmatrix} 0 & \eta\Omega \\ -\eta\Omega & 0 \end{pmatrix}, \quad \delta\mathbf{D} = \begin{pmatrix} \sigma+\rho & 0 \\ 0 & \sigma+\rho \end{pmatrix},$$

$$\mathbf{K} = \begin{pmatrix} -\alpha^2 & 0 \\ 0 & -\alpha^2 \end{pmatrix}, \quad \nu\mathbf{N} = \begin{pmatrix} 0 & \rho\Omega \\ -\rho\Omega & 0 \end{pmatrix}. \tag{33}$$

The system depends on the spin Ω and four parameters

$$\eta = \frac{I_a}{I_d}, \quad \sigma = \frac{b_s}{I_d}, \quad \rho = \frac{b_r}{I_d}, \quad \alpha^2 = \frac{mgL}{I_d}, \tag{34}$$

where α is the non-spinning pendulum frequency and η is responsible for the shape of the gyropendulum: for $\eta < 1$ the pendulum is prolate, and for $\eta > 1$ it is oblate. Parameters σ and ρ correspond to the stationary and rotating damping respectively. We notice that the stationary damping contributes only to the matrix $\delta\mathbf{D}$ while the rotating damping is responsible also for the appearance of the non-conservative positional forces described by the skew-symmetric matrix $\nu\mathbf{N}$. Thus, the Crandall gyropendulum can be treated as a conservative gyroscopic system perturbed by weak damping and non-conservative positional forces.

For $\sigma = \rho = 0$ the pendulum is stabilized by gyroscopic forces for $\Omega^2 > {\Omega_0}^2$. At the points of the stability boundary $\Omega = \pm\Omega_0$ the spectrum of the gyropendulum has a pair of double purely imaginary eigenvalues $\pm i\omega_0$, where according to (2) and (12) $\Omega_0 = 2\alpha/\eta$ and $\omega_0 = \alpha$. Writing the Liénard-Chipart conditions for the characteristic polynomial of the Crandall gyropendulum with the damping forces

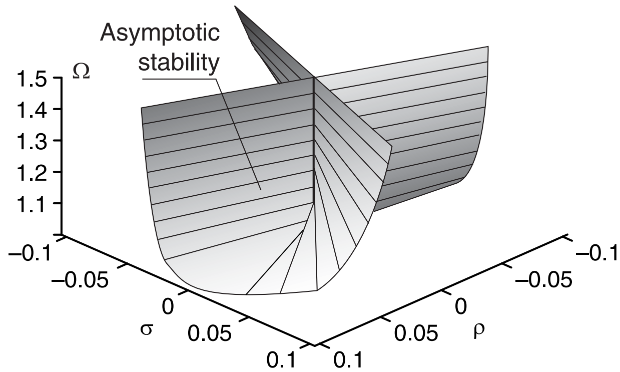

Fig. 3. The domain of the gyroscopic stabilization of the Crandall gyropendulum is a half of the Whitney umbrella.

we find the inequalities defining the asymptotic stability domain

$$\sigma + \rho > 0 \tag{35}$$

$$\eta^2\Omega^2 + (\sigma+\rho)^2 - 2\alpha^2 > 0 \tag{36}$$

$$\Omega^2 - \frac{(\sigma+\rho)^2\alpha^2}{\sigma\eta\rho + \eta\rho^2 - \rho^2} > 0. \tag{37}$$

Since the inequality (37) implies

$$\Omega^2 > {\Omega_0}^2 + \frac{1}{\rho}\frac{\alpha^2}{\eta^2}\frac{(\sigma\eta + \rho(\eta-2))^2}{\sigma\eta + \rho(\eta-1)} \geq {\Omega_0}^2, \tag{38}$$

the asymptotic stability domain is given only by the conditions (35) and (37), which can be written in the form

$$\Omega > \Omega_{cr}^{+}(\rho,\sigma), \ \ \Omega < \Omega_{cr}^{-}(\rho,\sigma), \ \ \sigma + \rho > 0, \tag{39}$$

where the critical values of the spin Ω as a function of the two damping parameters are

$$\Omega_{cr}^{\pm}(\rho,\sigma) = \pm\frac{(\sigma+\rho)\alpha}{\sqrt{-\rho^2+\rho^2\eta+\rho\eta\sigma}}. \tag{40}$$

Equations (40) describe two surfaces in the space of the parameters ρ, σ, and Ω. Both surfaces have Whitney umbrella singularities at the points $(0,0,\pm\Omega_0)$. The surface $\Omega_{cr}^{+}(\rho,\sigma)$ is shown in Fig 3. for $\alpha = 1$ and $\eta = 2$. The inequality (35) selects the stable pocket of the Whitney umbrella. In spite of the fact that the formulae for the critical spin analogous to (40) were found by Crandall with the use of a perturbation technique, the singular nature of the asymptotic stability domain was not recognized in [17].

As it follows from the expressions (38), $\Omega_{cr}^{+} \geq \Omega_0$ and $\Omega_{cr}^{-} \leq -\Omega_0$, which can be interpreted as the *destabilization* of the conservative gyroscopic system by the damping and non-conservative positional forces. The critical loads coincide only for the specific ratios of the coefficients of the stationary and rotating damping

$$\frac{b_s}{b_r} = \frac{\sigma}{\rho} = \frac{2-\eta}{\eta} = \frac{\Omega_0}{\omega_0} - 1 \tag{42}$$

in agreement with the result obtained in [17].

V. Conclusion

It was found that the boundary of the gyroscopic stabilization domain of a non-conservative system possesses the Whitney umbrella singularity. The singularity is responsible for the high sensitivity of the critical gyroscopic parameter to small variations of the matrices of damping and circulatory forces. The price for the gyroscopic stabilization of a non-conservative system is generally higher values of the gyroscopic parameter and non-trivial choice of balance of damping and non-conservative positional forces. Explicit analytical approximations of the boundary near the singularity and estimates of the critical gyroscopic parameter were derived in an analytic form for the systems with arbitrary even degrees of freedom. Finally, it was established that the stability boundary of the Crandall gyropendulum, considered as a mechanical example, consists of two pockets of two Whitney umbrellas.

Acknowledgements

The author thanks Peter Hagedorn and the Department of Mechanical Engineering of the Technical University of Darmstadt where the research for this paper has been done during the Alexander von Humboldt Fellowship.

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