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A NONSMOOTH OPTIMIZATION PROBLEM

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A linear autonomous nonconservative mechanical system is considered. The system smoothly depends on the nonconservative load parameter and on the vector of design parameters. The objective functional is defined as the minimal value of the nonconservative load parameter when the static or dynamic loss of stability takes place. The problem of finding the design parameters maximizing the objective functional is formulated. It is shown that such a problem of optimization is nonsmooth. The values of design parameters for which the objective functional is discontinuous are found. Necessary conditions of local extrema for such values are derived.

1. INTRODUCTION

Let us consider the following linear autonomous nonconservative mechanical system (damping and gyroscopic forces are absent):

$$\ddot{\mathbf{y}} + \mathbf{A}\mathbf{y} = 0. \tag{1}$$

Here $\mathbf{A}$ is a real nonsymmetric $m \times m$ matrix of positional nonconservative forces and $\mathbf{y}$ is an m-vector of generalized coordinates. The dots indicate the time differentiation. System (1) is said to be *circulatory* [1, 2]. If we seek a solution to system (1) in the form $\mathbf{y} = \mathbf{u}\exp\left(i\sqrt{\lambda}t\right)$ (here $\sqrt{\lambda}$ is the oscillation frequency), then we come to the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u}. \tag{2}$$

System (1) is stable if all the eigenvalues of (2) are positive and semisimple. If the spectrum of $\mathbf{A}$ contains a multiple eigenvalue with a Jordan chain or a complex eigenvalue, then we have the case of dynamic instability (flutter). Negative eigenvalues correspond to the case of static instability (divergence).

Let us assume that the matrix $\mathbf{A}$ smoothly depends on a vector of real parameters $\mathbf{p} \in \mathbb{R}^n$. As is known [3–5], the boundary of a stability domain for an n-parameter circulatory system is not smooth and may have singularities. In the case of general position, the boundary segments of codimension 1 correspond to those matrices $\mathbf{A}$ that contain either the simple zero eigenvalue or a positive double eigenvalue with a Jordan chain of length 2 (the other eigenvalues are positive and simple). Boundary singularities of the stability domain are of higher codimension and correspond to matrices with a more complicated Jordan structure [3, 5]. In the case of general position we have the situation when at interior points of the stability domain all eigenvalues of $\mathbf{A}$ are simple.

2. FORMULATION OF THE OPTIMIZATION PROBLEM

In the vector $\mathbf{p} \in \mathbb{R}^n$ we choose one of its components (for example, the first one). This component denoted by q will be called the *nonconservative load parameter*, whereas the other components denoted by $\mathbf{z} = (z_1, \dots, z_{n-1})$ will be called the *design parameters*. Hence, $\mathbf{p} = (q, \mathbf{z})$. Let us study the behavior of circulatory system (1), depending on $q \geq 0$ and $\mathbf{z}$. It is assumed that for $q = 0$ and for any $\mathbf{z}$ from an open domain $U \subset \mathbb{R}^{n-1}$ all eigenvalues of $\mathbf{A}(0, \mathbf{z})$ are positive and simple (i.e., $(0, \mathbf{z})$ is an interior point of the stability domain).

The one-parameter family $\mathbf{A}(q)$ of circulatory systems in general position contains matrices with simple spectra (only for some isolated values of q may the matrix $\mathbf{A}$ have either a Jordan block of order 2 with a double real eigenvalue or the simple zero eigenvalue [3, 4]). If the vector $\mathbf{z}$ is varied, then we may expect that matrices of a more complicated Jordan structure can appear in the family $\mathbf{A}(q)$ (among them, only matrices whose spectra contain multiple positive and zero eigenvalues correspond to the boundary of the stability domain). In order to describe the Jordan structure J, we use the notation given in [3] (for example, $J = \alpha^3 0$ means that the matrix $\mathbf{A}$ contains the simple zero eigenvalue and a Jordan block of third order with a triple real eigenvalue α; all the other eigenvalues are simple and nonzero).

Definition. Let us assume that the above vector $\mathbf{z} \subset U$ of design parameters is given. We say that the parameter q takes its *critical value* q_J^* if the spectrum of $\mathbf{A}(q_J^*, \mathbf{z})$ with a Jordan structure J consists of real eigenvalues $\lambda \geq 0$ among which there exists at least one zero eigenvalue of algebraic multiplicity $k \geq 1$ or at least one positive eigenvalue of algebraic multiplicity $k \geq 2$. The minimal element of the set $\Omega_{\mathbf{z}} = \{q_{J_1}^*, q_{J_2}^*, \ldots\}$ formed by all critical values of q is called the *critical load functional* and is denoted by $q^*(\mathbf{z})$.

The purpose of optimization of nonconservative system (1) according to the stability criterion is to widen the interval of safe (in the sense of stability loss) values of q with the aid of the proper choice of $\mathbf{z}$:

$$q^*(\mathbf{z}) \to \sup, \quad \mathbf{z} \in U \subset \mathbb{R}^{n-1}. \tag{3}$$

In problems of optimum design [6 – 10], the vector $\mathbf{z}$ usually corresponds to the mass distribution or to the stiffness of a structure. Additional restrictions may be imposed on $\mathbf{z}$ (for example, isoperimetric conditions). As is shown below, in our problem the main difficulty is that the functional is nonsmooth (therefore, we do not take into account such restrictions).

3. THE GRADIENT METHOD

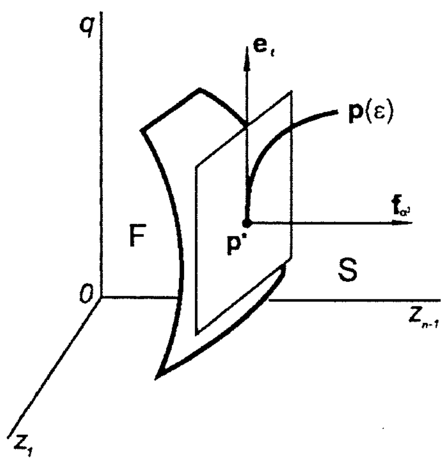

Fig. 1. The boundary between the domains of stability (S) and flutter (F)

As a rule, problem (3) is solved numerically; traditionally, the gradient method is used [6 – 10]. Let us consider a point $\mathbf{p}^* = (q_{\alpha^2}^*, \mathbf{z})$ belonging to the smooth part (a hypersurface) of the boundary between the stability and flutter domains (Figure 1). At this point of the boundary there exists a tangential plane such that an expression for the gradient of the critical load $q_{\alpha^2}^*$ follows from the equation of this plane. Such a tangential plane can be found if we consider the splitting of the double eigenvalue $\alpha > 0$ along the curves $\mathbf{p}(\varepsilon) = \mathbf{p}^* + \varepsilon \mathbf{e} + o(\varepsilon)$ issuing out of the point $\mathbf{p}^*$ in an arbitrary direction $\mathbf{e} \in \mathbb{R}^n$, where $\varepsilon \geq 0$ is a small parameter. Recall that the double eigenvalue α corresponds to the Jordan chain consisting of the eigenvector $\mathbf{u}_0$ and the associated vector $\mathbf{u}_1$:

$$\mathbf{A}\mathbf{u}_0 = \alpha \mathbf{u}_0, \quad \mathbf{A}\mathbf{u}_1 = \alpha \mathbf{u}_1 + \mathbf{u}_0.$$

In addition, α possesses the chain consisting of the eigenvector $\mathbf{v}_0$ and the associated vector $\mathbf{v}_1$ of the adjoint problem:

$$\mathbf{A}^T \mathbf{v}_0 = \alpha \mathbf{v}_0, \quad \mathbf{A}^T \mathbf{v}_1 = \alpha \mathbf{v}_1 + \mathbf{v}_0.$$

The vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$, and $\mathbf{v}_1$ are associated by the relations of orthogonality and normalization:

$$\mathbf{v}_0^T \mathbf{u}_0 = 0, \quad \mathbf{v}_0^T \mathbf{u}_1 \equiv \mathbf{v}_1^T \mathbf{u}_0 = 1.$$

In the case of general position, the splitting of α is described by the formula [5]

$$\lambda = \alpha \pm \sqrt{\varepsilon \langle \mathbf{f}_{\alpha^2}, \mathbf{e} \rangle} + O(\varepsilon). \tag{4}$$

Here the angular brackets $\langle , \rangle$ indicate the scalar product of two real vectors; the components of $\mathbf{f}_{\alpha^2} \in \mathbb{R}^n$ are specified by the relations

$$f_{\alpha^2}^1 = \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_0, \quad f_{\alpha^2}^s = \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial z_{s-1}} \mathbf{u}_0, \quad s = 2, \ldots, n. \tag{5}$$

Let us denote $\mathbf{r}_{\alpha^2} = (f_{\alpha^2}^2, \ldots, f_{\alpha^2}^n)$; then, $\mathbf{f}_{\alpha^2} = (f_{\alpha^2}^1, \mathbf{r}_{\alpha^2})$. Formula (4) is valid if the radicand is not equal to zero. When $\langle \mathbf{f}_{\alpha^2}, \mathbf{e} \rangle < 0$, α is split into two complex-conjugate eigenvalues (flutter). If $\langle \mathbf{f}_{\alpha^2}, \mathbf{e} \rangle > 0$, then the splitting yields two positive eigenvalues (stability). Hence, the vector $\mathbf{f}_{\alpha^2}$ is a vector normal to the boundary and belongs to the stability domain (Figure 1). For vectors $\mathbf{e}_t$ of the tangential plane, the following equality holds:

$$\langle \mathbf{f}_{\alpha^2}, \mathbf{e}_t \rangle = 0. \tag{6}$$

Substituting $\varepsilon \mathbf{e}_t = \Delta \mathbf{p} + o(\varepsilon)$ into (6) and taking into account (5), we obtain the equation describing the plane tangent to the stability boundary at the point $\mathbf{p}^*$:

$$\Delta q = \langle \mathbf{g}_{\alpha^2}, \Delta \mathbf{z} \rangle. \tag{7}$$

In equation (7), the vector

$$\mathbf{g}_{\alpha^2} = -\frac{\mathbf{r}_{\alpha^2}}{f^1_{\alpha^2}}, \quad f^1_{\alpha^2} \neq 0 \tag{8}$$

is the gradient of the critical load $q^*_{\alpha^2}$ with respect to the design parameters $\mathbf{z}$. Let us assume that $q^*(\mathbf{z}) = q^*_{\alpha^2}$ with $\mathbf{z}$ given. If $\mathbf{g}_{\alpha^2} \neq 0$, then the variation $\Delta \mathbf{z} = \gamma \mathbf{g}_{\alpha^2}$ is an improving one, since the increment $\Delta q^* = \gamma |\mathbf{g}_{\alpha^2}|^2$ is greater than zero for sufficiently small $\gamma > 0$.

The condition $\mathbf{g}_{\alpha^2} = 0$ is necessary for the vector $\mathbf{z}$ to be a solution to problem (3) in those domains of its variation where the objective functional smoothly depends on the design parameters. In another frequently occurring situation, the gradient significantly grows in such a way that for some $\mathbf{z}$ the functional is discontinuous and $\mathbf{g}_{\alpha^2} \to \infty$. Note that since the publication of [6] this phenomenon has been an insurmountable obstacle to solving the problems of optimization of circulatory systems: all attempts to find an improving variation leading away from a point of discontinuity or to prove the extremality of q^* at this point were unsuccessful (see, for example, [6–10]). Let us consider this case in more detail.

4. METAMORPHOSES OF CHARACTERISTIC CURVES $\lambda(q)$

Let us denote $q^*(\mathbf{z}) = q^*_{\alpha^2}$ for some $\mathbf{z}$ and assume that the gradient $\mathbf{g}_{\alpha^2}$ does not exist. According to (8), this is possible when $f^1_{\alpha^2} = 0$, i.e., when the normal to the stability boundary becomes orthogonal to the q-axis in the parameter space (Figure 1):

$$\mathbf{f}_{\alpha^2} = (0, \mathbf{r}_{\alpha^2}). \tag{9}$$

Recall that the point $\mathbf{p}^* = (q^*_{\alpha^2}, \mathbf{z})$ belongs to the boundary between the flutter and stability domains. A vector $\mathbf{e}_t \in \mathbb{R}^n$ tangent to this boundary may be chosen in the form

$$\mathbf{e}_t = (1, 0, \ldots, 0). \tag{10}$$

Let us consider the splitting of a double eigenvalue α along the curves $\mathbf{p}(\varepsilon) = \mathbf{p}^* + \varepsilon \mathbf{e}_t + \varepsilon^2 \mathbf{d} + o(\varepsilon^2)$ tangent to the boundary with $\varepsilon = 0$. In this case, by virtue of condition (6) the leading term of the series expansion of the perturbed eigenvalue in small parameter is of order ε [5]:

$$\lambda = \alpha + \varepsilon \mu + o(\varepsilon). \tag{11}$$

The coefficient μ can be determined from the quadratic equation [5]

$$\mu^2 - 2\langle \mathbf{h}_{\alpha^2}, \mathbf{e}_t \rangle \mu + \langle \mathbf{H}_{\alpha^2} \mathbf{e}_t, \mathbf{e}_t \rangle = \langle \mathbf{f}_{\alpha^2}, \mathbf{d} \rangle. \tag{12}$$

The components of the real vector $\mathbf{h}_{\alpha^2}$ and the elements of the real symmetric matrix $\mathbf{H}_{\alpha^2}$ are expressed in terms of the first and second derivatives of $\mathbf{A}$ with respect to the parameters and in terms of the vectors $\mathbf{u}_0$, $\mathbf{v}_0$, $\mathbf{u}_1$, and $\mathbf{v}_1$. In particular, we have [5]

$$2h^1_{\alpha^2} = \mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_0 + \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_1, \quad H^{11}_{\alpha^2} = -\frac{1}{2} \mathbf{v}_0^T \frac{\partial^2 \mathbf{A}}{\partial q^2} \mathbf{u}_0 + \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{G}_0 \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_0, \tag{13}$$

where the matrix $\mathbf{G}_0 = [\mathbf{A}_0 - \alpha \mathbf{I} - \mathbf{v}_0 \mathbf{v}_0^T]^{-1}$ is nonsingular and $\mathbf{I}$ is the identity matrix. Substituting (11) into (12) and taking into account (9) and (10), we come to the conclusion that the splitting of α along the curves $\mathbf{p}(\varepsilon)$ is described by the formula

$$\left(\lambda - \alpha - h^1_{\alpha^2} \frac{dq}{d\varepsilon} \varepsilon\right)^2 = \left\langle \mathbf{r}_{\alpha^2}, \frac{1}{2} \frac{d^2 \mathbf{z}}{d\varepsilon^2} \varepsilon^2 \right\rangle - H^{11}_{\alpha^2} \left(\frac{dq}{d\varepsilon} \varepsilon\right)^2 + \left(h^1_{\alpha^2} \frac{dq}{d\varepsilon} \varepsilon\right)^2 + o(\varepsilon^2), \tag{14}$$

where all the derivatives are taken at $\varepsilon = 0$. By virtue of (10), increments of q and $\mathbf{z}$ along the curve $\mathbf{p}(\varepsilon)$ tangent to the stability boundary at the point $\mathbf{p}^*$ are of the form

$$\Delta q = \left.\frac{dq}{d\varepsilon}\right|_{\varepsilon=0} \varepsilon + o(\varepsilon), \quad \Delta \mathbf{z} = \frac{1}{2}\left.\frac{d^2\mathbf{z}}{d\varepsilon^2}\right|_{\varepsilon=0} \varepsilon^2 + o(\varepsilon^2). \tag{15}$$

Using (15), we can rewrite equation (14) as

$$\left(\Delta\lambda - h^1_{\alpha^2}\Delta q\right)^2 - D(\Delta q)^2 = \langle \mathbf{r}_{\alpha^2}, \Delta\mathbf{z}\rangle + o((\Delta q)^2), \tag{16}$$

where $\Delta\lambda = \lambda - \alpha$, $\Delta q = q - q^*_{\alpha^2}$, $D = (h^1_{\alpha^2})^2 - H^{11}_{\alpha^2}$, and the coefficients $h^1_{\alpha^2}$ and $H^{11}_{\alpha^2}$ are specified by (13).

Equation (16) describes the splitting of the double eigenvalue α when the load parameter and the vector of design parameters are varied in a neighborhood of the point $\mathbf{p}^* = (q^*_{\alpha^2}, \mathbf{z})$. On the other hand, this equation approximately describes the behavior of two characteristic curves $\lambda(q)$ when the vector $\mathbf{z}$ is varied. Depending on the sign of D, there exist the following two possibilities: if $D > 0$, then the section of the flutter domain by the plane of the vectors $\mathbf{f}_{\alpha^2}$ and $\mathbf{e}_t$ is convex (Figure 1), whereas if $D < 0$, then the section of the stability domain is convex.

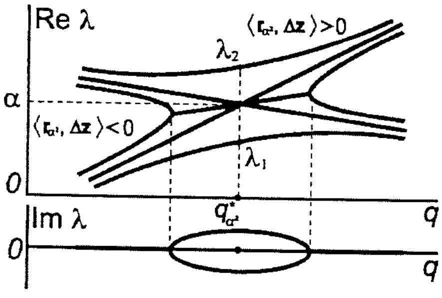

Fig. 2. Metamorphosis of characteristic curves

We restrict our consideration to the case $D > 0$. If the vector of design parameters is fixed ($\Delta\mathbf{z} = 0$), then the double eigenvalue α is split (according to (16)) into two simple positive eigenvalues; these eigenvalues linearly change with Δq (Figure 2), as should be the case when $f^1_{\alpha^2} = 0$. If $\Delta\mathbf{z}$ satisfies the condition

$$\langle \mathbf{r}_{\alpha^2}, \Delta\mathbf{z}\rangle > 0, \tag{17}$$

then two nonintersecting curves $\lambda(q)$ are solutions to equation (16); these curves are the branches of the hyperbola corresponding to simple eigenvalues λ (the case of stability). When $\langle \mathbf{r}_{\alpha^2}, \Delta\mathbf{z}\rangle < 0$, positive eigenvalues are arranged along two branches of an adjacent hyperbola; in this case, however, there exists an interval of changes in q where α is split into a complex-conjugate pair (the case of flutter; Figure 2). Thus, equation (16) describes the metamorphosis of two curves $\lambda(q)$; in the case of general position, this metamorphosis takes place in a neighborhood of the point $(q^*_{\alpha^2}, \mathbf{z})$ when the gradient $\mathbf{g}_{\alpha^2}$ tends to infinity [6–10].

5. IMPROVING VARIATIONS AND NECESSARY CONDITIONS FOR EXTREMUM IN THE CASE OF DISCONTINUITY OF THE FUNCTIONAL

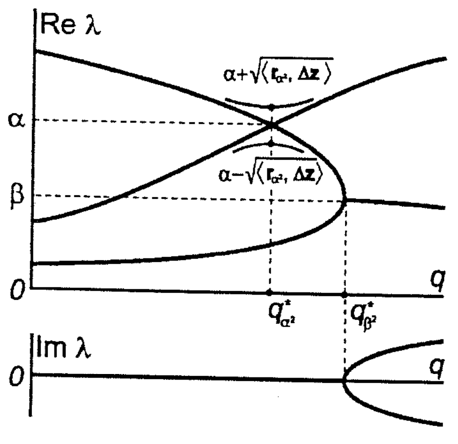

Fig. 3. Discontinuity of the critical load functional

Suppose we are given a vector $\mathbf{z}$ and the first two minimal values $q^*_{\alpha^2}, q^*_{\beta^2} \in \Omega_{\mathbf{z}}$ corresponding to double eigenvalues α and β, respectively ($q^*_{\alpha^2} < q^*_{\beta^2}$). It is assumed that $\mathbf{g}_{\alpha^2} \to \infty$ ($f^1_{\alpha^2} = 0$) and $\mathbf{g}_{\beta^2} \neq 0$ ($f^1_{\beta^2} \neq 0$). Since $q^*_{\alpha^2} < q^*_{\beta^2}$, the critical load functional $q^*(\mathbf{z})$ is equal to $q^*_{\alpha^2}$ (Figure 3). This value is not maximal, because it follows from equation (16) that any small variation $\Delta\mathbf{z}$ satisfying inequality (17) causes a step-wise increase in the functional. Indeed, under condition (17) equation (16) moves the characteristic curves $\lambda(q)$ apart at the point $q = q^*_{\alpha^2}$ (this means that the critical value $q^*_{\alpha^2}$ disappears; Figure 2). In the case of general position, the critical value $q^*_{\beta^2}$ and its gradient $\mathbf{g}_{\beta^2}$ slightly change. The value of the functional at the point $\mathbf{z} + \Delta\mathbf{z}$ exceeds its value at the point $\mathbf{z}$ by a quantity of order $q^*_{\beta^2} - q^*_{\alpha^2}$. At the same time, the former may be less than the critical value $q^*_{\beta^2}$ computed at the point $\mathbf{z}$ if $q^*_{\beta^2}$ is the supremum of the functional q^* in a neighborhood of the point $\mathbf{z}$. Let us find necessary conditions which are to be valid if it is known that at the point $\mathbf{z}$ we have

$$q^*(\mathbf{z}) = q^*_{\alpha^2}, \quad f^1_{\alpha^2} = 0, \quad q^*_{\alpha^2} < q^*_{\beta^2}, \quad f^1_{\beta^2} \neq 0, \quad \mathbf{g}_{\beta^2} \neq 0. \tag{18}$$

First we assume that the critical value $q^*_{\beta^2}$ is not a supremum. Then, this value can be increased if we move along the gradient $\mathbf{g}_{\beta^2}$ and prevent the frequency curves from converging. Hence, the improving variation $\Delta\mathbf{z}$ must satisfy condition (17) and the inequality

$$\Delta q^*_{\beta^2} = \langle \mathbf{g}_{\beta^2}, \Delta\mathbf{z}\rangle > 0. \tag{19}$$

We shall look for this variation in the form of the following linear combination:

$$\Delta\mathbf{z} = c_1 \mathbf{r}_{\alpha^2} + c_2 \mathbf{g}_{\beta^2}. \tag{20}$$

The meaning of $2\sqrt{\langle \mathbf{r}_{\alpha^2}, \Delta\mathbf{z}\rangle}$ is simple: it is equal to the difference between the two eigenvalues $\lambda_{1,2} = \alpha \pm \Delta\lambda$ (the double eigenvalue α is split into these eigenvalues at the point $(q^*_{\alpha^2}, \mathbf{z} + \Delta\mathbf{z})$, see Figure 2). Let the distance between λ_1 and λ_2 be equal to $2\sqrt{\varepsilon_1}$ ($\varepsilon_1 > 0$) and the increment $\Delta q^*_{\beta^2}$ be equal to $\varepsilon_2 > 0$. Substituting variation (20) into conditions (17) and (19), we obtain the following equations for the coefficients c_1 and c_2:

$$\begin{aligned} c_1\langle \mathbf{r}_{\alpha^2}, \mathbf{r}_{\alpha^2}\rangle + c_2\langle \mathbf{r}_{\alpha^2}, \mathbf{g}_{\beta^2}\rangle &= \varepsilon_1, \\ c_1\langle \mathbf{g}_{\beta^2}, \mathbf{r}_{\alpha^2}\rangle + c_2\langle \mathbf{g}_{\beta^2}, \mathbf{g}_{\beta^2}\rangle &= \varepsilon_2. \end{aligned} \tag{21}$$

For arbitrary right-hand sides, system (21) has a solution if and only if its determinant is not equal to zero. Since the matrix of this system is the Gramian matrix of the vectors $\mathbf{r}_{\alpha^2}$ and $\mathbf{g}_{\beta^2}$, its determinant is not equal to zero if these vectors are linearly independent. Variation (20) causes the critical load $q^*_{\beta^2}$ to increase, preventing the characteristic curves from overlapping when $q < q^*_{\beta^2}$. Thus, the following lemma is valid.

Lemma. *If the above vectors $\mathbf{r}_{\alpha^2}$ and $\mathbf{g}_{\beta^2}$ are linearly independent, then for any small $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ there exists an improving variation of form* (20) *whose coefficients c_1 and c_2 are solutions to system* (21).

When the determinant of the above Gramian matrix is equal to zero, the vectors $\mathbf{r}_{\alpha^2}$ and $\mathbf{g}_{\beta^2}$ become linearly dependent. In this case, system (21) may have a solution only for special values of ε_1 and ε_2 when the right-hand side of (21) is orthogonal to the solution of the homogeneous system obtained from (21). This condition of solvability can easily be written down in explicit form as the following relation between ε_1 and ε_2:

$$\varepsilon_2 = \varepsilon_1 \frac{\langle \mathbf{r}_{\alpha^2}, \mathbf{g}_{\beta^2}\rangle}{\langle \mathbf{r}_{\alpha^2}, \mathbf{r}_{\alpha^2}\rangle}. \tag{22}$$

When the numerator in (22) is positive, a solution to system (21) exists with $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$; hence, an improving variation also exists. If $\mathbf{r}_{\alpha^2}$ and $\mathbf{g}_{\beta^2}$ satisfy the condition

$$\langle \mathbf{r}_{\alpha^2}, \mathbf{g}_{\beta^2}\rangle < 0, \tag{23}$$

then ε_1 and ε_2 are opposite in sign and variation (20) with c_1 and c_2 determined from (21) is not improving. Indeed, in this case an increase of the critical load $q^*_{\beta^2}$ ($\varepsilon_2 > 0$) causes a metamorphosis of frequency curves ($\varepsilon_1 < 0$) and a step-wise decrease of the functional; our attempt to move the frequency curves apart ($\varepsilon_1 > 0$) at the point $q^*_{\alpha^2}$ causes a decrease of the critical load $q^*_{\beta^2}$ ($\varepsilon_2 < 0$). Now we formulate necessary conditions of local extrema in problem (3) for the vectors $\mathbf{z}$ on which the gradient $\mathbf{g}_{\alpha^2}$ of the minimal critical value of $q^*_{\alpha^2}$ tends to infinity (Figures 1 and 3).

Theorem. *Let conditions* (18) *be valid for a vector* $\mathbf{z}$. *In order for $q^*_{\beta^2}$ to be the supremum of the functional $q^*(\mathbf{z})$ in a small neighborhood of* $\mathbf{z}$, *it is necessary that the vectors $\mathbf{r}_{\alpha^2}$ and $\mathbf{g}_{\beta^2}$ be linearly dependent:* $\mathbf{r}_{\alpha^2} = \gamma\mathbf{g}_{\beta^2}$ *with* $\gamma < 0$.

The proof of the theorem follows from the above lemma and inequality (23). In conclusion we note that conditions (18) describe the situation of general position for a system with two or more parameters. For this reason, a discontinuity of the critical load functional may often be observed when solving the problems of optimization of circulatory systems [6 – 10]. The above theorem allows us to find an extremum or to move away from the point of discontinuity of the functional q^*.

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