On Krein space related perturbation theory for MHD $\alpha^2$-dynamos

Oleg N. Kirillov$^{*1}$ and Uwe Günther$^{**2}$

1 Moscow State Lomonosov University, Michurinskii pr. 1, 119192 Moscow, Russia
2 Research Center Rossendorf, POB 510119, D-01314 Dresden, Germany

The spectrum of the spherically symmetric $\alpha^2$-dynamo is studied in the case of idealized boundary conditions. Starting from the exact analytical solutions of models with constant $\alpha$-profiles a perturbation theory and a Galerkin technique are developed in a Krein-space approach. With the help of these tools a very pronounced $\alpha$-resonance pattern is found in the deformations of the spectral mesh as well as in the unfolding of the diabolical points located at the nodes of this mesh. Non-oscillatory as well as oscillatory dynamo regimes are obtained. An estimation technique is developed for obtaining the critical $\alpha$-profiles at which the eigenvalues enter the right spectral half-plane with non-vanishing imaginary components (at which overcritical oscillatory dynamo regimes form).

1 Resonant deformation of the spectral mesh of the mean field MHD $\alpha^2$-dynamo

The mean field $\alpha^2$-dynamo of magnetohydrodynamics (MHD) [1] plays a similarly paradigmatic role in MHD dynamo theory like the harmonic oscillator in quantum mechanics. In its kinematic regime this dynamo is described by a linear induction equation for the magnetic field. For spherically symmetric models. Due to the fundamental symmetry of its differential expression, the potentials $\alpha$-profiles $\alpha(r)$ the vector of the magnetic field can be decomposed into poloidal and toroidal components and expanded in spherical harmonics. After additional time separation, the induction equation reduces to a set of $l$-decoupled boundary eigenvalue problems [1, 2]

$$\mathfrak{A}_\alpha u = \lambda u, \quad u(r \to 0) = u(1) = 0$$

(1)

for matrix differential operators

$$\mathfrak{A}_\alpha := \begin{pmatrix} -A_l & \alpha(r) \\ A_{l,\alpha} & -A_l \end{pmatrix}, \quad A_l := -\partial_r^2 + \frac{l(l+1)}{r^2}, \quad A_{l,\alpha} := -\partial_r \alpha(r) \partial_r + \alpha(r) \frac{l(l+1)}{r^2} = \alpha(r) A_l - \alpha'(r) \partial_r.$$ (2)

The boundary conditions in (1) are idealized ones and formally coincide with those for dynamos in a high conductivity limit of the dynamo maintaining fluid/plasma. We will restrict our subsequent considerations to this case and assume a domain

$$D(\mathfrak{A}_\alpha) = \left\{ u \in \mathring{H} = L_2(0,1) \oplus L_2(0,1) \mid u(r \to 0) = u(1) = 0 \right\}$$

(3)

in the Hilbert space $(\mathring{H}, \langle \cdot, \cdot \rangle)$. The $\alpha$-profile $\alpha(r)$ is a smooth real function $C^2(0,1) \ni \alpha(r) : (0,1) \to \mathbb{R}$. It plays the role of the potential in dynamo models. Due to the fundamental symmetry of its differential expression, the operator $\mathfrak{A}_\alpha$ is a symmetric operator in a Krein space $(\mathfrak{K}, [\cdot,\cdot])$ with indefinite inner product $[\cdot,\cdot] = (J,\cdot)$ and for the chosen domain (3) it is also selfadjoint in this space $[\mathfrak{A}_\alpha x,y] = [x,\mathfrak{A}_\alpha y], \quad x,y \in \mathfrak{K}$. For constant $\alpha$-profiles $\alpha(r) \equiv \alpha_0 =$ const, $r \in [0,1)$, the spectrum and eigenvectors of the operator matrix $\mathfrak{A}_\alpha$ are

$$\lambda_\pm \equiv \lambda_\pm^n(\alpha_0) = -\rho_n \pm \alpha_0 \sqrt{\rho_n} \in \mathbb{R}, \quad n \in \mathbb{Z}^+, \quad u_\pm^n = \begin{pmatrix} 1 \\ \pm \sqrt{\rho_n} \end{pmatrix} u_n \in \mathbb{R}^2 \oplus L_2(0,1),$$

(5)

and correspond to Krein space states of positive and negative type $[u_m^+, u_n^+] = \pm 2 \sqrt{\rho_m \rho_n} \delta_{mn}, \quad [u_m^-, u_n^-] = 0, \quad u_m^\pm \in \mathfrak{K}_\pm \subset \mathfrak{K}$. The functions $u_m$ in (5) are Riccati-Bessel functions

$$u_m(r) = N_m r^{1/2} J_{l+\frac{1}{2}}(\sqrt{\rho_m} r), \quad N_m := \frac{\sqrt{2}}{J_{l+\frac{1}{2}}(\sqrt{\rho_m})}, \quad (u_m, u_n) = \delta_{mn}, \quad \|u_n\| = 1.$$ (6)

Accordingly, the coefficients $\rho_n > 0$ in (5) are the squares of Bessel function zeros $J_{l+\frac{1}{2}}(\sqrt{\rho_n}) = 0, \quad 0 < \sqrt{\rho_1} < \sqrt{\rho_2} < \cdots$. The branches $\lambda_\pm^n$ of the spectrum are real-valued linear functions of the parameter $\alpha_0$ with slopes $\pm \sqrt{\rho_n}$ and form a meshlike structure in the $(\alpha_0, \Re \lambda)$-plane, as depicted in Fig. 1a. The nodes of the spectral mesh in Fig. 1a are the intersection

* Corresponding author: e-mail: kirillov@imec.msu.ru
** E-mail: u.guenther@fz-rossendorf.de
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