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Dissipation induced instabilities in continuous non-conservative systems

Oleg N. Kirillov[*1] and **Alexander P. Seyranian** [1]

[1] Institute of Mechanics, Moscow State Lomonosov University, Michurinskii pr. 1, 119192 Moscow, Russia

In this contribution we analyze the stabilizing and destabilizing effect of small damping for rather general class of continuous non-conservative systems, described by the non-self-adjoint boundary eigenvalue problems. Explicit asymptotic expressions obtained for the stability domain allow us to predict when a given combination of the damping parameters leads to increase or to decrease in the critical non-conservative load. The results obtained explain why different types of internal and external damping so surprisingly influence on the stability of non-conservative systems. As a mechanical example the stability of a viscoelastic rod with small internal and external damping, loaded by tangential follower force, is studied in detail.

1 Weakly damped continuous non-conservative systems

We consider the boundary eigenvalue problem arising in stability problems for viscoelastic systems

$$N(q)u + \lambda D(\mathbf{k})u + \lambda^2 Mu = 0, \quad \mathbf{U}_N(q)\mathbf{u} + \lambda\mathbf{U}_D(\mathbf{k})\mathbf{u} + \lambda^2\mathbf{U}_M\mathbf{u} = 0. \tag{1}$$

The coefficients of the linear differential operators N, D, and M of order m, and of the matrices $\mathbf{U}_N$, $\mathbf{U}_D$, and $\mathbf{U}_M$ of dimension $m \times 2m$ are assumed to be real. The vector $\mathbf{u} = (u(0), u'_x(0), \dots, u_x^{(m-1)}(0), u(1), u'_x(1), \dots, u_x^{(m-1)}(1))$ of dimension $2m$ consists of the values of the function $u(x)$ and its derivatives evaluated at the boundary points $x = 0$ and $x = 1$. The operator N and the matrix $\mathbf{U}_N$ smoothly depend on the real "load" parameter $q \geq 0$. The coefficients of the differential operator D and of the matrix $\mathbf{U}_D$ are smooth functions of the vector of real "damping" parameters $\mathbf{k} = (k_1, \dots, k_{n-1})$, and $\mathbf{k} = 0$ yields $D(0) = 0$, $\mathbf{U}_D(0) = 0$. The operator M and the matrix $\mathbf{U}_M$ are parametrically independent.

Let the unperturbed circulatory system

$$N(q)u + \lambda^2 Mu = 0, \quad \mathbf{U}_N(q)\mathbf{u} + \lambda^2\mathbf{U}_M\mathbf{u} = 0 \tag{2}$$

have discrete spectrum for $0 \leq q < q_0$, consisting of simple purely imaginary eigenvalues λ (stability). For increasing load parameter q two simple purely imaginary eigenvalues move along the imaginary axis until they collide at $q{=}q_0$ forming the double eigenvalue $i\omega_0$ with the Keldysh chain [1], consisting of an eigenfunction $u_0(x)$ and associated function $u_1(x)$. After the collision the eigenvalues diverge in the directions perpendicular to the imaginary axis of the complex plane (flutter instability). Such a scenario is known as the strong interaction of eigenvalues and is a typical mechanism of the loss of stability for circulatory systems [2]. All remaining eigenvalues $\pm i\omega_{0,j}$, $\omega_{0,j} > 0$ of the unperturbed system at $q = q_0$ are assumed to be simple and purely imaginary. Therefore, in the absence of dissipative forces ($\mathbf{k} = 0$) the value $q = q_0$ is the boundary between the stability and flutter domains. Introduction of small damping ($\mathbf{k}{\neq}0$) changes the instability mechanism. With the variation of the parameter q the eigenvalues generally move separately on the complex plane without interaction; one of the two eigenvalues is in the left side of the complex plane, while another one crosses the imaginary axis and goes to the right side at $q = q_{cr}(\mathbf{k})$. Analysis of bifurcation of the double eigenvalue due to small dissipative perturbation gives an approximation of the critical load $q_{cr}(\mathbf{k})$

$$q_{cr}(\mathbf{k}) = q_0 + \frac{(\langle\mathbf{f},\mathbf{k}\rangle + \langle\mathbf{H}\mathbf{k},\mathbf{k}\rangle)^2}{\widetilde{f}\langle\mathbf{h},\mathbf{k}\rangle^2} - \frac{\omega_0^2}{\widetilde{f}}\langle\mathbf{G}\mathbf{k},\mathbf{k}\rangle, \tag{3}$$

where the angular brackets denote the inner product of vectors in R^{n-1}. The components of the real vector $\mathbf{f}$ and the real quantity $\widetilde{f}$ are expressed by means of the eigenfunctions and associated functions of the adjoint boundary eigenvalue problems and the derivatives of the differential operators and the matrices of the boundary conditions

$$\widetilde{f} = \int_0^1 \bar{v}_0(x)\frac{\partial N}{\partial q}u_0(x)dx + \mathbf{v}_0^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_N}{\partial q}\mathbf{u}_0, \quad f_r = \int_0^1 \bar{v}_0(x)\frac{\partial D}{\partial k_r}u_0(x)dx + \mathbf{v}_0^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0, \tag{4}$$

where the asterisk denotes Hermitian conjugation. Similarly, one can find the real vector $\mathbf{h}$ and the real matrices $\mathbf{H}$ and $\mathbf{G}$.

As it follows from equation (3), the function $q_{cr}(\mathbf{k})$ is singular at the point $\mathbf{k}{=}0$, and the critical load as a function of $n{-}1$ variable has no limit as $\mathbf{k}{=}(k_1, \dots, k_{n-1})$ tends to zero. However, there exists the limit $\lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}})$ for any

* Corresponding author: e-mail: kirillov@imec.msu.ru, Phone: +7 095 939 2039, Fax: +7 095 939 0165

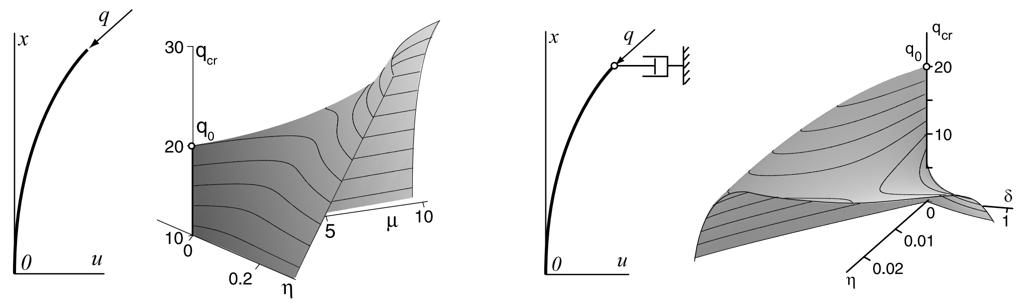

Fig. 1 Stability domains (6), (8) of a viscoelastic rod in a viscous medium (left) and with a dash-pot at the free end (right)

direction $\widetilde{\mathbf{k}}$ such that $\langle \mathbf{h}, \widetilde{\mathbf{k}} \rangle \neq 0$. Substituting $\mathbf{k}{=}\epsilon\widetilde{\mathbf{k}}$ into equation (3) and taking the limit one can find an explicit expression approximating *the jump in the critical load* due to small damping. For $\langle \mathbf{f}, \widetilde{\mathbf{k}} \rangle{=}0$ the jump in the critical load does not happen since $\lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}}){=}q_0$. According to equation (3) the critical load in this case tends to q_0 as $\epsilon \to 0$. However, the function $q_{cr}(\epsilon\widetilde{\mathbf{k}})$ can decrease for $\langle \mathbf{f}, \widetilde{\mathbf{k}} \rangle{=}0$, if $\langle \mathbf{G}\widetilde{\mathbf{k}}, \widetilde{\mathbf{k}} \rangle < 0$. For $\langle \mathbf{G}\widetilde{\mathbf{k}}, \widetilde{\mathbf{k}} \rangle > 0$ the critical load is increasing $q_{cr}(\epsilon\widetilde{\mathbf{k}}) \geq q_0$ (stabilization).

2 Stability of a viscoelastic rod with different types of external damping

Consider transverse vibrations of a cantilevered rod about vertical equilibrium position. The rod is made of the viscoelastic Kelvin-Voight material with the damping coefficient $\eta \geq 0$. It is assumed that the rod is loaded by the tangential follower force q at its free end, as shown in Figure 1. If the rod is vibrating in a viscous medium with the damping coefficient $\mu \geq 0$, then the investigation of stability of this system is reduced to the study of the boundary eigenvalue problem [3]

$$(1+\eta\lambda)u''''_{xxxx} + qu''_{xx} + (\lambda^2+\mu\lambda)u = 0, \quad u(0){=}0,\ u'_x(0){=}0,\ u''_{xx}(1){=}0,\ u'''_{xxx}(1){=}0. \tag{5}$$

When the damping is absent ($\eta = \mu = 0$), the elastic rod is stable for the follower loads in the interval $0 \leq q < q_0$, where $q_0{=}20.05$ [4]. At $q = q_0$ the spectrum of the problem is discrete and consists of the pair of the double eigenvalues $\pm i\omega_0$ ($\omega_0{=}11.02$), other eigenvalues $\pm i\omega_{0,s}$, $s = 1, 2, \ldots$ being simple and purely imaginary. Combining the stability conditions given by simple and double eigenvalues we find that the viscoelastic rod in viscous medium is asymptotically stable in the vicinity of the point $\eta = 0$, $\mu = 0$, $q = q_0$, if the following inequalities are satisfied

$$\eta > 0,\ \mu > -157.9\eta,\ q < q_{cr}(\eta,\mu); \quad q_{cr}(\eta,\mu){=}q_0 - \frac{1902\eta^2}{(14.34\eta + 0.091\mu)^2} + 12.68\eta\mu + 0.053\mu^2. \tag{6}$$

Approximation of the asymptotic stability domain (6) in the space of the parameters η, μ, q is shown in Figure 1, which clearly shows the existence of the region where $q_{cr}(\eta,\mu) > q_0$. Hence, small internal (η) and external (μ) damping can stabilize the viscoelastic rod loaded by the follower force.

If we neglect the influence of the damping due to resistance of the medium and assume that a dash-pot with the damping coefficient δ is attached to the free end of the rod, then we arrive to the boundary eigenvalue problem [5]

$$(1+\eta\lambda)u''''_{xxxx} + qu''_{xx} + \lambda^2 u = 0, \quad u(0){=}u'_x(0){=}0,\ u''_{xx}(1){=}(1{+}\eta\lambda)u'''_{xxx}(1){-}\delta\lambda u(1){=}0. \tag{7}$$

Approximation of the stability domain for this problem is described by the conditions

$$\eta > 0,\ \delta > -107.0\eta,\ q < q_{cr}(\eta,\delta); \quad q_{cr}(\eta,\delta){=}q_0 - \frac{(43.61\eta{+}0.719\delta)^2}{(14.34\eta{+}0.134\delta)^2} - 1368\eta^2 + 248.8\delta\eta. \tag{8}$$

According to (8) the critical load decreases in a discontinuous manner for any combination of positive damping parameters η and δ in the vicinity of the origin, Figure 1. Thus, small external damping caused by a dash-pot destabilizes the rod, contrary to the resistance of a medium, which has a stabilizing effect.

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