Pergamon
www.elsevier.com/locate/jappmathmech
PII: S0021-8928(02)00046-1
J. Appl. Maths Mechs, Vol. 66, No. 3, pp. 371–385, 2002

0021-8928/02/\$—see front matter

METAMORPHOSES OF CHARACTERISTIC CURVES IN CIRCULATORY SYSTEMS†

O. N. KIRILLOV and A. P. SEYRANIAN

Moscow

(*Received 6 September* 2001)

Explicit formulae are developed for typical metamorphoses of characteristic curves in circulatory systems which depend on a vector of parameters. The formulae obtained utilize information on the system only at the point of confluence of the curves and enable one to analyse both qualitatively and quantitatively the behaviour of the oscillation frequencies in the neighbourhood of that point. Quadratic approximations are found for the domains of flutter and divergence, and the relation between the convexity properties of these domains and the type of metamorphosis of the characteristic curves is established. © 2002 Elsevier Science Ltd. All rights reserved.

In circulatory systems which depend on parameters, one often observes a phenomenon of overlapping of the characteristic curves [1–5]. The curves describing the dependence of the eigenvalues of the linear operator of the system on a specified parameter – often the non-conservative load parameter – approach one another as the other parameters are varied, coming together at a certain point, and are then modified, forming a closed curve of complex eigenvalues – an "instability bubble". A description of this phenomenon will be given below.

1. STATEMENT OF THE PROBLEM

Consider the oscillations of a linear autonomous non-conservative mechanical system when there are no damping or gyroscopic forces

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\mathbf{q} = 0 \tag{1.1}$$

where $\mathbf{M} = \mathbf{M}^T > 0$ and $\mathbf{C} \neq \mathbf{C}^T$ are real $m \times m$ matrices of inertial coefficients and non-conservative positional forces, $\mathbf{q}$ is the vector of generalized coordinates, of dimension m, and the dots denote differentiation with respect to time t. System (1.1) is frequently referred to as a circulatory system [6, 7]. Seeking solutions of system (1.1) in the form $\mathbf{q} = \mathbf{u}e^{i\omega t}$, where ω is the oscillation frequency, and using the notation $\mathbf{A} = \mathbf{M}^{-1}\mathbf{C}$, $\lambda = \omega^2$, we arrive at the eigenvalue problem

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u} \tag{1.2}$$

The inverse matrix $\mathbf{M}^{-1}$ exists because $\mathbf{M}$ is symmetric and positive-definite. That the matrix $\mathbf{C}$ is non-symmetric implies that the matrix $\mathbf{A}$ in (1.2) is also non-symmetric. Hence the spectrum of problem (1.2) may contain complex eigenvalues λ.

System (1.1) is stable if all the eigenvalues of problem (1.2) are positive and the number of eigenvectors belonging to each λ is equal to the algebraic multiplicity of λ as a root of the characteristic equation. In other words, simple elementary divisors correspond to each positive eigenvalue. If all the λ's are real, but some of them are negative, then system (1.1) is statically unstable (divergence). The presence of complex eigenvalues λ implies oscillatory instability (flutter).

We shall assume that the matrices $\mathbf{M}$ and $\mathbf{C}$, hence also $\mathbf{A}$, are smooth functions of a parameter vector $\mathbf{p} \in \mathbb{R}^n$. It is well known [8–10] that in the case of the general position the smooth parts of the boundary of the stability domain – briefly, the stability boundary – of a circulatory system are made up of surfaces of codimension 1, at whose points the matrix $\mathbf{A}$ contains either a simple eigenvalue zero or a positive two-fold eigenvalue with a Jordan chain of length 2, all other eigenvalues λ being positive and simple. Generally speaking, the stability boundary is not smooth and may have singularities of higher codimension, corresponding to matrices whose Jordan normal form has a more complicated structure [8, 10].

†*Prikl. Mat. Mekh.* Vol. 66, No. 3, pp. 381–395, 2002.

As a rule, the stability boundary of a system depending on several parameters cannot be determined explicitly, while a numerical solution proves to be laborious. Nevertheless, there are problems where one has to determine the critical value of only one specified parameter p_j $(1 \leqslant j \leqslant n)$, fixing the other parameters p_s $(s \neq j)$. In such cases one is interested in the dependence of the eigenvalues or oscillation frequencies, considered in the space (Re λ, Im λ, p_j) or (Re ω, Im ω, p_j), on this parameter [1–5]. The function $\lambda(p_j)$ will be called a *characteristics curve* and the function $\omega(p_j)$ a *frequency curve*.

In the case of the general position, the curves $\lambda(p_j)$ and $\omega(p_j)$ lie either in the real plane or outside it, corresponding to complex eigenvalues or frequencies. When λ and ω leave the real plane, they may "collide" with other eigenvalues of frequencies to form a two-fold eigenvalue λ or frequency ω with a Jordan chain of length 2. If the parameter p_j moves from a stability domain into a divergence domain, the characteristic curve will only change sign, remaining in the real plane, but the frequency curve will go from the real plane to the imaginary plane, through formation of a two-fold zero frequency with a Jordan chain of length 2 [9]. Variation of the parameters p_s $(s \neq j)$ leads to deformation of the characteristic curve $\lambda(p_j)$ and frequency curve $\omega(p_j)$. The families thus obtained contain curves with more complicated behaviour, in a non-removable manner. For example, two curves may approach one another and merge when the parameters are varied, but then metamorphose in such a way that they undergo a qualitative change of shape.

We shall investigate some typical metamorphoses of characteristic curves and frequency curves in n-parameter circulatory systems near the smooth parts of the stability boundary.

Theorem 1. Let $\lambda(p_j)$ be a characteristic curve of system (1.1). Suppose that, at a non-singular point $\mathbf{p}_0 = (p_{0,1}, \dots, p_{0,n})^T$ of the boundary between the stability and divergence domains, the matrix $\mathbf{A}(\mathbf{p}_0)$ contains a simple eigenvalue $\lambda_0 = 0$ with right eigenvector $\mathbf{u}_0$ and left eigenvector $\mathbf{v}_0$, $\lambda(p_{0,j}) = 0$ and the following condition holds

$$\mathbf{v}_0^T\mathbf{A}_{,j}\mathbf{u}_0\,|_{\mathbf{p}=\mathbf{p}_0} = 0, \qquad \mathbf{A}_{,j} = \partial\mathbf{A}/\partial p_j \tag{1.3}$$

Then the behaviour of the characteristic curve $\lambda(p_j)$ in the neighbourhood of the point (Re λ_0, Im λ_0, $p_{0,j}$) is described by the equation

$$\Delta\lambda + [\mathbf{v}_0^T\mathbf{A}_{,j}\mathbf{G}_0\mathbf{A}_{,j}\mathbf{u}_0 - \tfrac{1}{2}\mathbf{v}_0^T\mathbf{A}_{,jj}\mathbf{u}_0]\Delta p_j^2 = \sum_{s=1,s\neq j}^{n}\mathbf{v}_0^T\mathbf{A}_{,s}\mathbf{u}_0\Delta p_s \tag{1.4}$$

$$\Delta\lambda = \lambda - \lambda_0, \quad \Delta p_k = p_k - p_{0,k}, \quad \mathbf{G}_0 = [\mathbf{A}(\mathbf{p}_0) - \lambda_0\mathbf{I} - \bar{\mathbf{v}}_0\mathbf{v}_0^T]^{-1}, \quad A_{,jj} = \partial^2A/\partial p_j^2$$

where $\mathbf{I}$ is the identity matrix.

Theorem 2. Let $\lambda'(p_j)$ and $\lambda''(p_j)$ be two characteristic curves of system (1.1). Suppose that at a non-singular point $\mathbf{p}_0 = (p_{0,1}, \dots, p_{0,n})^T$ of the boundary between the stability and flutter domains the matrix $\mathbf{A}(\mathbf{p}_0)$ contains a two-fold eigenvalue $\lambda_0 > 0$ to which correspond right and left Jordan chains of vectors, $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$, respectively, $\lambda'(p_{0,j}) = \lambda''(p_{0,j}) = \lambda_0$ and condition (1.3) holds. Then these characteristic curves are modified in the neighbourhood of the point (Re λ_0, Im λ_0, $p_{0,j}$) as described by the following equation

$$\left(\Delta\lambda - \frac{1}{2}[\mathbf{v}_1^T\mathbf{A}_{,j}\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{A}_{,j}\mathbf{u}_1]\Delta p_j\right)^2 - \\ -\left(\frac{1}{4}[\mathbf{v}_1^T\mathbf{A}_{,j}\mathbf{u}_0 + \mathbf{v}_0^T\mathbf{A}_{,j}\mathbf{u}_1]^2 - \mathbf{v}_0^T\left[\mathbf{A}_{,j}\mathbf{G}_0\mathbf{A}_{,j} - \frac{1}{2}\mathbf{A}_{,jj}\right]\mathbf{u}_0\right)\Delta p_j^2 = \sum_{s=1,s\neq j}^{n}\mathbf{v}_0^T\mathbf{A}_{,s}\mathbf{u}_0\Delta p_s \tag{1.5}$$

Formulae defining the behaviour of the corresponding frequency curves are obtained from Eqs (1.4) and (1.5) by making the substitution $\lambda = \omega^2$.

Theorem 1 and 2 will be proved in Section 4 using perturbation theory for the eigenvalues of non-selfadjoint operators.

2. PERTURBATIONS OF EIGENVALUES

Suppose that at a point $\mathbf{p}_0 \in \mathbb{R}^n$ the spectrum of the matrix $\mathbf{A}(\mathbf{p}_0)$ contains an eigenvalue λ_0. Considering a smooth one-parameter curve issuing from the point $\mathbf{p}_0$, say $\mathbf{p}(\varepsilon)$, $\varepsilon \geqslant 0$, let us expand the function

$\mathbf{p}(\varepsilon)$ in Taylor series in the neighbourhood of $\varepsilon = 0$ (the prime denotes differentiation with respect to ε)

$$\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon\mathbf{p}' + \tfrac{1}{2}\varepsilon^2\mathbf{p}'' + o(\varepsilon^2) \tag{2.1}$$

Since $\mathbf{A}$ is a smooth function of the parameter vector, the function $\mathbf{A}(\mathbf{p}(\varepsilon))$ may also be represented in series form

$$\mathbf{A}(\mathbf{p}(\varepsilon)) = \mathbf{A}(\mathbf{p}_0) + \varepsilon\sum_{s=1}^{n}\mathbf{A}_{,s}p_s' + \frac{1}{2}\varepsilon^2\left(\sum_{s=1}^{n}\mathbf{A}_{,s}p_s'' + \sum_{s,t=1}^{n}\mathbf{A}_{,st}p_s'p_t'\right) + o(\varepsilon^2) \tag{2.2}$$

The derivatives of $\mathbf{A}$ with respect to the parameters are evaluated at the point $\mathbf{p}_0$ and may be expressed in terms of the corresponding derivatives of the matrices $\mathbf{M}$ and $\mathbf{C}$. For example

$$\mathbf{A}_{,s} = \mathbf{M}_0^{-1}\mathbf{C}_{,s} - \mathbf{M}_0^{-1}\mathbf{M}_{,s}\mathbf{M}_0^{-1}\mathbf{C}_0, \qquad \mathbf{M}_0 = \mathbf{M}(\mathbf{p}_0), \qquad \mathbf{C}_0 = \mathbf{C}(\mathbf{p}_0)$$

Let $\mathbf{A}_k$ denote the coefficient of ε^k in expansion (2.2), and let us define vectors

$$\mathbf{e} = \mathbf{p}'\big|_{\varepsilon=0}, \qquad \mathbf{d} = \frac{1}{2}\mathbf{p}''\Big|_{\varepsilon=0}; \qquad \mathbf{e}, \mathbf{d} \in \mathbb{R}^n \tag{2.3}$$

Then

$$\mathbf{A}_0 = \mathbf{A}(\mathbf{p}_0), \qquad \mathbf{A}_1 = \sum_{s=1}^{n}\mathbf{A}_{,s}e_s, \qquad \mathbf{A}_2 = \sum_{s=1}^{n}\mathbf{A}_{,s}d_s + \frac{1}{2}\sum_{s,t=1}^{n}\mathbf{A}_{,st}e_se_t \tag{2.4}$$

When the vector $\mathbf{p}_0$ is varied as in (2.1), the eigenvalue λ_0 and the eigenvector $\mathbf{u}_0$ receive increments which may be expressed as series in integer or fractional powers of ε, depending on the Jordan structure corresponding to λ_0.

If λ_0 is a simple eigenvalue of $\mathbf{A}_0$ with eigenvector $\mathbf{u}_0$, the perturbed eigenvalue λ and eigenvector $\mathbf{u}$ are smooth functions of ε and may be expressed as Taylor series [11]

$$\lambda = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \dots, \qquad \mathbf{u} = \mathbf{u}_0 + \varepsilon\mathbf{w}_1 + \varepsilon^2\mathbf{w}_2 + \dots \tag{2.5}$$

Substituting expansions (2.2) and (2.5) into Eq. (1.2) and equating terms with like powers of ε, we obtain an eigenvalue problem for the unperturbed matrix $\mathbf{A}_0$

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{u}_0 = 0 \tag{2.6}$$

and equations determining the first and second corrections to the eigenvalue λ_0 and eigenvector $\mathbf{u}_0$

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_1 = \lambda_1\mathbf{u}_0 - \mathbf{A}_1\mathbf{u}_0 \tag{2.7}$$

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_2 = \lambda_1\mathbf{w}_1 - \mathbf{A}_1\mathbf{w}_1 + \lambda_2\mathbf{u}_0 - \mathbf{A}_2\mathbf{u}_0 \tag{2.8}$$

Let $(\mathbf{a}, \mathbf{b}) = a_1\bar{b}_1 + \dots + a_m\bar{b}_m$ denote the scalar product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{C}^m$. Together with problem (2.6), let us consider the adjoint eigenvalue problem

$$(\mathbf{A}_0^T - \bar{\lambda}_0\mathbf{I})\mathbf{v}_0 = 0 \tag{2.9}$$

We require the vector $\mathbf{v}_0$ to satisfy the following normalization condition

$$(\mathbf{u}_0, \mathbf{v}_0) = 1 \tag{2.10}$$

Given the vector $\mathbf{u}_0$, condition (2.10) enables one to determine the vector $\mathbf{v}_0$ uniquely. For the perturbed vector $\mathbf{u}$ we use the normalization condition

$$(\mathbf{u}, \mathbf{v}_0) = 1 \tag{2.11}$$

which uniquely determines all the terms λ_i and $\mathbf{w}_i$ in expansions (2.5).

Equations (2.7) and (2.8) are solvable if and only if their right-hand sides are orthogonal to the solution of the homogeneous adjoint eigenvalue problem (2.9). Hence, also using the normalization conditions (2.10) and (2.11), we obtain expressions for the coefficients λ_1 and λ_2 in expansion (2.5)

$$\lambda_1 = (\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0), \quad \lambda_2 = (\mathbf{A}_2\mathbf{u}_0, \mathbf{v}_0) + (\mathbf{A}_1\mathbf{w}_1, \mathbf{v}_0) \tag{2.12}$$

The vector $\mathbf{w}_1$ is found from Eq. (2.17) using the operator $\mathbf{G}_0$ inverse to $\mathbf{A}_0 - \lambda_0\mathbf{I}$

$$\mathbf{w}_1 = \mathbf{G}_0(\lambda_1\mathbf{u}_0 - \mathbf{A}_1\mathbf{u}_0)$$

The first equality of (2.12) is a necessary and sufficient condition for the existence of the inverse operator. Since det $(\mathbf{A}_0 - \lambda_0\mathbf{I}) = 0$, the matrix $(\mathbf{A}_0 - \lambda_0\mathbf{I})^{-1}$ does not exist. At the same time, the operator $\mathbf{G}_0$ may be represented using a non-singular matrix in the form [12]

$$\mathbf{G}_0 = [\mathbf{A}_0 - \lambda_0\mathbf{I} - \bar{\mathbf{v}}_0\mathbf{v}_0^T]^{-1} \tag{2.13}$$

If λ_0 is a two-fold eigenvalue with Jordan chain of length 2, an eigenvector $\mathbf{u}_0$ and an associated vector $\mathbf{u}_1$ exist satisfying the equations

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{u}_0 = 0 \quad (\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{u}_1 = \mathbf{u}_0 \tag{2.14}$$

as well as an eigenvector and an associated vector of the adjoint system

$$(\mathbf{A}_0^T - \bar{\lambda}_0\mathbf{I})\mathbf{v}_0 = 0 \quad (\mathbf{A}_0^T - \bar{\lambda}_0\mathbf{I})\mathbf{v}_1 = \mathbf{v}_0 \tag{2.15}$$

The vectors $\mathbf{u}_0$, $\mathbf{u}_1$, $\mathbf{v}_0$ and $\mathbf{v}_1$ satisfy the following orthogonality and normalization conditions

$$(\mathbf{u}_0, \mathbf{v}_0) = 0 \quad (\mathbf{u}_1, \mathbf{v}_0) \equiv (\mathbf{u}_0, \mathbf{v}_1) = 1 \tag{2.16}$$

When the parameter vector (2.1) is varied, the perturbed eigenvalue with Jordan chain of length 2 and its eigenvector are represented by series in powers of the small parameter $\varepsilon^{1/2}$ [11]

$$\lambda = \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + \ldots, \qquad \mathbf{u} = \mathbf{u}_0 + \varepsilon^{1/2}\mathbf{w}_1 + \varepsilon\mathbf{w}_2 + \varepsilon^{3/2}\mathbf{w}_3 + \ldots \tag{2.17}$$

It is convenient to subject $\mathbf{u}$ to the normalization condition

$$(\mathbf{u}, \mathbf{v}_1) = 1 \tag{2.18}$$

It follows from condition (2.18) and expansions (2.17) that $(\mathbf{w}_i, \mathbf{v}_1) = 0$ $(i = 1, 2, \ldots)$. Substituting expressions (2.2) and (2.17) into eigenvalue problem (1.2) and equating terms with like powers of ε, we obtain equations for the corrections to λ_0 and $\mathbf{w}_0$

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_1 = \lambda_1\mathbf{u}_0 \tag{2.19}$$

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_2 = -\mathbf{A}_1\mathbf{u}_0 + \lambda_1\mathbf{w}_1 + \lambda_2\mathbf{u}_0 \tag{2.20}$$

$$(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_4 = -\mathbf{A}_1\mathbf{w}_2 - \mathbf{A}_2\mathbf{u}_0 + \lambda_1\mathbf{w}_3 + \lambda_2\mathbf{w}_2 + \lambda_3\mathbf{w}_1 + \lambda_4\mathbf{u}_0 \tag{2.21}$$

An expression for the vector $\mathbf{w}_1$ satisfying normalization condition (2.18) follows easily from Eq. (2.19)

$$\mathbf{w}_1 = \lambda_1(\mathbf{u}_1 - \mathbf{u}_0(\mathbf{u}_1, \mathbf{v}_1)) \tag{2.22}$$

Equation (2.2) is solvable if and only if its right-hand side is orthogonal to the vector $\mathbf{v}_0$. By (2.16) and (2.22), this condition may be written in the form

$$\lambda_1^2 = (\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0) \tag{2.23}$$

The vector $\mathbf{w}_2$ is found from Eq. (2.20) using the operator $\mathbf{G}_0$ of (2.13)

$$\mathbf{w}_2 = \mathbf{G}_0(\lambda_1\mathbf{w}_1 + \lambda_2\mathbf{u}_0 - \mathbf{A}_1\mathbf{u}_0) \tag{2.24}$$

If the right-hand side of Eq. (2.23) does not vanish, there are two non-trivial solutions

$$\lambda_1 = \pm\sqrt{(\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0)}$$

to which the vectors $\mathbf{w}_1$ of (2.22) correspond.

Let us consider the degenerate case

$$(\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_0) = 0 \tag{2.25}$$

Then the perturbed eigenvector $\mathbf{u}$ and eigenvalue λ are determined by terms of the order of ε, that is, by the coefficients $\mathbf{w}_2$ and λ_2, since now in (2.17) we have

$$\lambda_1 = 0, \qquad \mathbf{w}_1 = 0 \tag{2.26}$$

In order to find the coefficient λ_2, we write down the condition for Eq. (2.21) to be solvable. Taking the normalization condition (2.16) and degeneracy condition (2.6) into consideration, we obtain

$$\lambda_2(\mathbf{w}_2, \mathbf{v}_0) - (\mathbf{A}_1\mathbf{w}_2, \mathbf{v}_0) - (\mathbf{A}_2\mathbf{u}_0, \mathbf{v}_0) = 0 \tag{2.27}$$

Multiplying both sides of Eq. (2.20) by $\mathbf{v}_1$, we arrive at the relation

$$(\mathbf{w}_2, \mathbf{v}_0) = \lambda_2 - (\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_1) \tag{2.28}$$

Substituting expressions (2.24) and (2.28) into Eq. (2.27), we find that in the degenerate case (2.25) the coefficient λ_2 is determined by a quadratic equation

$$\lambda_2^2 - \lambda_2[(\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_1) + (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_0)] - (\mathbf{A}_2\mathbf{u}_0, \mathbf{v}_0) + (\mathbf{G}_0(\mathbf{A}_1\mathbf{u}_0), \mathbf{A}_1^T\mathbf{v}_0) = 0 \tag{2.29}$$

Remark. If the discriminant of Eq. (2.29) does not vanish, this guarantees bifurcation of the two-fold eigenvalue λ_0 in the degenerate case (2.25). Using the explicit form of the solution of Eq. (2.29), we can write the bifurcation condition in equivalent form as

$$2\lambda_2 \neq (\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_1) + (\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_0) \tag{2.30}$$

We shall prove that if conditions (2.25) and (2.30) hold simultaneously, then all the odd coefficients λ_{2i-1}, $\mathbf{w}_{2i-1}$ of expansions (2.17) vanish. The proof is by induction. Putting $i = 1$ in (2.26), we get $\lambda_1 = 0$, $\mathbf{w}_1 = 0$. Now suppose that for some integer $k > 1$

$$\lambda_{2i-1} = 0, \quad \mathbf{w}_{2i-1} = 0, \quad i = 1, \ldots, k \tag{2.31}$$

Consider the equations obtained by equating terms with powers $\varepsilon^{(2k+1)/2}$ and $\varepsilon^{(2k+3)/2}$, after substituting expansions (2.2) and (2.17) into problem (1.2)

$$\begin{aligned}(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_{2k+1} &= -\sum_{j=1}^{k}\mathbf{A}_j\mathbf{w}_{2(k-j)+1} + \sum_{i=1}^{2k}\lambda_i\mathbf{w}_{2k+1-i} + \lambda_{2k+1}\mathbf{u}_0 \\ (\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_{2k+3} &= -\sum_{j=1}^{k+1}\mathbf{A}_j\mathbf{w}_{2(k-j)+3} + \sum_{i=1}^{2k+2}\lambda_i\mathbf{w}_{2k+3-i} + \lambda_{2k+3}\mathbf{u}_0\end{aligned} \tag{2.32}$$

By the induction hypothesis (2.31), Eqs (2.32) may be simplified as follows:

$$\begin{aligned}(\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_{2k+1} &= \lambda_{2k+1}\mathbf{u}_0 \\ (\mathbf{A}_0 - \lambda_0\mathbf{I})\mathbf{w}_{2k+3} &= -\mathbf{A}_1\mathbf{w}_{2k+1} + \lambda_2\mathbf{w}_{2k+1} + \lambda_{2k+1}\mathbf{w}_2\lambda_{2k+3}\mathbf{u}_0\end{aligned} \tag{2.33}$$

By the first equation of (2.33)

$$\mathbf{w}_{2k+1} = \lambda_{2k+1}(\mathbf{u}_1 - \mathbf{u}_0(\mathbf{u}_1, \mathbf{v}_1)) \tag{2.34}$$

Multiplying both sides of the second equation of (2.33) by $\mathbf{v}_0$ and using relations (2.28) and (2.34), we obtain

$$\lambda_{2k+1}(2\lambda_2 - [(\mathbf{A}_1\mathbf{u}_1, \mathbf{v}_0) + (\mathbf{A}_1\mathbf{u}_0, \mathbf{v}_1)]) = 0 \tag{2.35}$$

Since condition (2.30) is assumed to hold, it follows from (2.35) and (2.34) that

$$\lambda_{2k+1} = 0, \qquad \mathbf{w}_{2k+1} = 0$$

We have thus proved that all the odd coefficients in expansions (2.17) vanish if conditions (2.25) and (2.30) hold simultaneously.

3. QUADRATIC APPROXIMATION OF THE STABILITY BOUNDARY

Thus, we have constructive formulae which enables us to compute the first and second corrections to the eigenvalues and eigenvectors, for perturbation of a non-conservative system, in terms of the derivatives of its matrices with respect to the parameters, and also in terms of the eigenvectors and associated vectors. In what follows we shall be interested only in real eigenvalues λ_0, since it is these that define the limits of stability. In that case the corresponding eigenvectors and associated eigenvectors may also be chosen to be real. We now define real vectors

$$\begin{aligned} &\mathbf{f} = (f_1,\dots,f_n), \qquad \mathbf{h} = (h_1,\dots,h_n); \\ &f_s = (\mathbf{A}_{,s}\mathbf{u}_0,\mathbf{v}_0), \qquad 2h_s = (\mathbf{A}_{,s}\mathbf{u}_0,\mathbf{v}_1) + (\mathbf{A}_{,s}\mathbf{u}_1,\mathbf{v}_0), \qquad s = 1,\dots,n \end{aligned} \tag{3.1}$$

as well as a real symmetric $n \times n$ matrix $\mathbf{H}$ with elements

$$\begin{aligned} &2H_{jk} = -(\mathbf{A}_{,kj}\mathbf{u}_0,\mathbf{v}_0) + (\mathbf{G}_0(\mathbf{A}_{,k}\mathbf{u}_0),\mathbf{A}_{,j}^T\mathbf{v}_0) + (\mathbf{G}_0(\mathbf{A}_{,j}\mathbf{u}_0),\mathbf{A}_{,k}^T\mathbf{v}_0) \\ &j,k = 1,\dots,n \end{aligned} \tag{3.2}$$

and we let $\langle \mathbf{a}, \mathbf{b}\rangle = a_1b_1 + \dots + a_nb_n$ denote the scalar product of vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. With this notation, using formulae (2.4), we have

$$\begin{aligned} &(\mathbf{A}_1\mathbf{u}_0,\mathbf{v}_0) = \langle \mathbf{f},\mathbf{e}\rangle, \qquad (\mathbf{A}_1\mathbf{u}_1,\mathbf{v}_0) + (\mathbf{A}_1\mathbf{u}_0,\mathbf{v}_1) = 2\langle \mathbf{h},\mathbf{e}\rangle \\ &-(\mathbf{A}_2\mathbf{u}_0,\mathbf{v}_0) + (\mathbf{G}_0(\mathbf{A}_1\mathbf{u}_0),\mathbf{A}_1^T\mathbf{v}_0) = \langle \mathbf{He},\mathbf{e}\rangle - \langle \mathbf{f},\mathbf{d}\rangle \end{aligned} \tag{3.3}$$

The vector $\mathbf{f} \neq 0$ has a simple geometrical meaning, namely, the normal to the stability boundary in its smooth parts. This is easily established by using (3.3) to rewrite the coefficients of expansions (2.5) and (2.17) for a simple zero eigenvalue and a two-fold positive eigenvalue

$$\lambda = \langle \mathbf{f},\mathbf{e}\rangle\varepsilon + o(\varepsilon), \qquad \lambda = \lambda_0 \pm \sqrt{\langle \mathbf{f},\mathbf{e}\rangle\varepsilon} + o(\varepsilon^{1/2}) \tag{3.4}$$

If the vector $\mathbf{e}$ belongs to the half-space defined by the inequality $\langle \mathbf{f}, \mathbf{e}\rangle > 0$, a slight variation of the parameters along the curve $\mathbf{p}(\varepsilon)$ will make a zero eigenvalue become positive; a two-fold eigenvalue λ_0 will bifurcate into two simple positive eigenvalues, implying stability. But $\langle \mathbf{f}, \mathbf{e}\rangle < 0$, a zero eigenvalue will become negative (divergence) and a two-fold eigenvalue λ_0 will bifurcate into a complex-conjugate pair (flutter). Consequently, evaluated at a non-singular point $\mathbf{p}_0$ of the stability boundary, the vector $\mathbf{f}$ of (3.1) will point into the stability domain (S) along the normal to the boundary (Fig. 1). The stability boundary itself is approximated in the neighbourhood of the point $\mathbf{p}_0$, to a first approximation, by its tangent plane

$$\langle \mathbf{f},\Delta\mathbf{p}\rangle = 0, \qquad \Delta\mathbf{p} = \mathbf{p} - \mathbf{p}_0 \tag{3.5}$$

Formulae (3.4) hold provided that $\langle \mathbf{f}, \mathbf{e}\rangle \neq 0$, which means that the curve $\mathbf{p}(\varepsilon)$ is transversal to the stability boundary. In the single-parameter family of matrices $\mathbf{A}(\mathbf{p}(\varepsilon))$ this is the case of general position, for which the behaviour of the characteristic curve $\lambda(\varepsilon)$ described by formulae (3.4) is typical [9].

The geometrical meaning of the vector $\mathbf{h}$ and the matrix $\mathbf{H}$ defined by (3.1) and (3.2) is related to quadratic approximation of the stability domain. In order to establish this meaning, consider the curves

$$\mathbf{p}(\varepsilon) = \mathbf{p}_0 + \varepsilon\mathbf{e}_* + \varepsilon^2\mathbf{d} + o(\varepsilon^2) \tag{3.6}$$

which are tangent to the stability boundary at $\varepsilon = 0$, so that one has the orthogonality condition

$$\langle \mathbf{f},\mathbf{e}_*\rangle = 0 \tag{3.7}$$

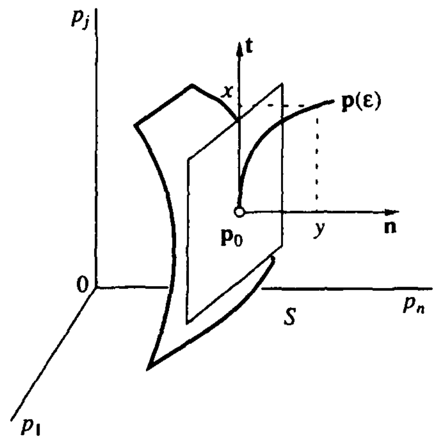

Fig. 1

Let $\mathbf{p}_0$ be a point on the boundary between the stability and divergence domains. We shall study the behaviour of the simple zero eigenvalue that defines this boundary along such a curve. By condition (3.7), the coefficient λ_1 in series (2.5) will vanish, and the expansion of the perturbed eigenvalue will begin with the ε^2 term. Transforming the right-hand side of formula (2.12) for λ_2 using (2.3) and (3.3), we obtain a formula for the increment of the simple zero eigenvalue

$$\lambda = \tfrac{1}{2}\langle \mathbf{f}, \mathbf{p}'' \rangle \varepsilon^2 - \langle \mathbf{H}\mathbf{p}', \mathbf{p}' \rangle \varepsilon^2 + o(\varepsilon^2) \tag{3.8}$$

Similarly, considering a point on the boundary between the stability and flutter domains and using relations (2.17), (2.26) and (2.29), we find that the bifurcation of the two-fold eigenvalue λ_0 along curves tangent to the stability boundary is defined by the equation

$$(\lambda - \lambda_0 - \langle \mathbf{h}, \mathbf{p}' \rangle \varepsilon)^2 = \tfrac{1}{2}\langle \mathbf{f}, \mathbf{p}'' \rangle \varepsilon^2 - \langle \mathbf{H}\mathbf{p}', \mathbf{p}' \rangle \varepsilon^2 + \langle \mathbf{h}, \mathbf{p}' \rangle^2 \varepsilon^2 + o(\varepsilon^2) \tag{3.9}$$

Note that condition (3.7) is also satisfied by curves on the boundary of the stability domain itself. Along such curves a simple zero eigenvalue remains equal to zero and a two-fold eigenvalue does not bifurcate. A necessary condition for this to occur is that the right-hand sides of formulae (3.8) and (3.9) should vanish. Writing these conditions in explicit form and taking into consideration that on these curves

$$\langle \mathbf{f}, \Delta\mathbf{p} \rangle = \tfrac{1}{2}\langle \mathbf{f}, \mathbf{p}'' \rangle \varepsilon^2 + o(\varepsilon^2) \tag{3.10}$$

we conclude that the stability boundary of the circulatory system (1.1) in the neighbourhood of a non-singular point is described up to $o(\|\Delta\mathbf{p}\|^2)$ by

$$F_k(\Delta\mathbf{p}) \equiv \langle \mathbf{f}, \Delta\mathbf{p} \rangle - G_k(\Delta\mathbf{p}) = 0, \quad k = 1, 2 \tag{3.11}$$

$$G_1(\Delta\mathbf{p}) = \langle \mathbf{H}\Delta\mathbf{p}, \Delta\mathbf{p} \rangle, \quad G_2(\Delta\mathbf{p}) = \langle (\mathbf{H} - \mathbf{h}\mathbf{h}^T)\Delta\mathbf{p}, \Delta\mathbf{p} \rangle$$

for $k = 1$, if the stability domain borders on the divergence domain, and for $k = 2$, if it has a common boundary with the flutter domain.

It follows from Eq. (3.9) that the flutter domain is defined by the inequality $F_2 < 0$, and the stability domain, accordingly, is defined by the inequality $F_2 > 0$. Considering vectors $\Delta\mathbf{p}$ belonging to the tangent plane (3.5), we deduce from the condition $F_2 > 0$ that if the quadratic form $G_2(\Delta\mathbf{p})$ is negative (positive) definite in the set (3.5), then the stability domain is concave (convex) and the flutter domain is convex (concave), since the tangent plane is contained in the stability (flutter) domain. Using Eq. (3.8) and reasoning in a similar way, we conclude that if the quadratic form $G_1(\Delta\mathbf{p})$ is negative (positive) definite in the set (3.5), the divergence domain is convex (concave) and the stability domain is concave (convex).

In particular, if the matrix $\mathbf{H}$ evaluated at a non-singular point of the stability boundary is negative definite, the stability domain is concave at that point, while the unstable (flutter or divergence) domain is convex.

Note that quadratic approximations of the stability boundaries of conservative systems have been obtained before [13].

4. METAMORPHOSES OF CHARACTERISTIC CURVES NEAR THE STABILITY BOUNDARY

Formulae (3.4) show what happens to the eigenvalues of the matrix of the non-conservative system (1.1) for transverse crossing of the stability boundary. Let us consider the question of how the eigenvalues behave along a straight line tangent to the stability boundary at a non-singular point $\mathbf{p}_0$, and in general along straight lines parallel to the tangent at a short distance from the boundary. The answer is given by the following lemma.

Lemma. Let $\mathbf{p}_0$ be a non-singular point on the stability boundary, let $\mathbf{f}$ be the normal vector (3.1) evaluated at the point, let $\mathbf{n} = \mathbf{f}/|\mathbf{f}|$ be the corresponding unit vector, and let $\mathbf{t} = \mathbf{e}_*/|\mathbf{e}_*|$ be the unit vector of an arbitrary tangent at the point $\mathbf{p}_0$. Then the section of the stability domain by the plane Oxy spanned by the vectors $\mathbf{n}$ and $\mathbf{t}$, with origin at the point $\mathbf{p}_0$, is described by one of the two inequalities

$$y > Px^2/|f|, \quad y > Qx^2/|f|$$
$$P = \langle \mathbf{Ht}, \mathbf{t}\rangle, Q = P - R^2, R = \langle \mathbf{h}, \mathbf{t}\rangle \tag{4.1}$$

depending on whether the stability domain borders on the divergence domain or the flutter domain. Under these conditions, the behaviour of a simple zero eigenvalue is governed by the equation

$$\lambda + Px^2 = y|\mathbf{f}| + o(x^2) \tag{4.2}$$

while the formula

$$(\lambda - \lambda_0 - Rx)^2 + Qx^2 = y|\mathbf{f}| + o(x^2) \tag{4.3}$$

describes bifurcation of a two-fold eigenvalue $\lambda_0 > 0$ with Jordan chain of length 2.

Proof. The behaviour of the zero and two-fold eigenvalues defining the stability boundary along the curves (3.6) is described by formulae (3.8) and (3.9). The curve (3.6) in the plane of the normal and tangent vectors may be expressed as

$$\mathbf{p}(\varepsilon) = \mathbf{p}_0 + x(\varepsilon)\mathbf{t} + y(\varepsilon)\mathbf{n} \tag{4.4}$$

where $x(\varepsilon)$ and $y(\varepsilon)$ are smooth functions of ε. By definition (2.3)

$$\mathbf{e}_* \equiv \mathbf{p}' = x'\mathbf{t} + y'\mathbf{n}, \quad 2\mathbf{d} \equiv \mathbf{p}'' = x''\mathbf{t} + y''\mathbf{n} \tag{4.5}$$

where all the derivatives are evaluated at $\varepsilon = 0$. It follows from relations (4.5) that

$$x'|_{\varepsilon=0} = |\mathbf{e}_*|, \quad y'|_{\varepsilon=0} = 0, \quad y''|_{\varepsilon=0} = 2\langle \mathbf{n}, \mathbf{d}\rangle \tag{4.6}$$

Using expressions (4.5) and (4.6) for the derivatives, we can write formulae (3.8) and (3.9) in the form

$$\lambda = -P(x'\varepsilon)^2 + \tfrac{1}{2}y''|\mathbf{f}|\varepsilon^2 + o(\varepsilon^2) \tag{4.7}$$

$$(\lambda - \lambda_0 - Rx'\varepsilon)^2 + Q(x'\varepsilon)^2 = \tfrac{1}{2}y''|\mathbf{f}|\varepsilon^2 + o(\varepsilon^2) \tag{4.8}$$

Since the first differential of the function $y(\varepsilon)$ vanishes at $\varepsilon = 0$, its increment is determined by the term of order ε^2. Consequently, formulae (4.2) and (4.3) hold in the neighbourhood of the point $\mathbf{p}_0$ in the plane of the vectors $\mathbf{n}$ and $\mathbf{t}$. The condition for λ to be positive as the parameters x and y vary, applied to Eq. (4.2), and the non-bifurcation condition, applied to Eq. (4.3), yield the second-order approximations (4.1) to the section of the stability domain by the plane Oxy.

Corollary. Theorems 1 and 2 follow immediately from the lemma. Indeed, condition (1.3) implies that the jth component of the normal vector $\mathbf{f}$ vanishes at the point $\mathbf{p}_0$. Since we are interested in the functions $\lambda(p_j)$, we have to take the unit tangent vector $\mathbf{t}$ parallel to the p_j axis (Fig. 1). All components of this vector vanish, except for $t_j = 1$. Under these conditions, obviously

$$\Delta p_j = x, \quad y|\mathbf{f}| = \sum_{s=1, s\neq j}^{n} f_s \Delta p_s$$

In addition, because of the special form of the unit tangent vector **t**, the scalar products in formulae (4.1)–(4.3) degenerate and contain only the components H_{jj} and h_j of the matrix **H** and vector **h**. Taking all this into consideration in (4.2) and (4.3) and expressing H_{jj}, h_j and f_s in terms of the derivatives of the matrix **A** with respect to the parameters and the vectors of the Jordan chain, we finally obtain formulae (1.4) and (1.5).

For a given unit tangent vector **t**, Eqs (4.2) and (4.3) describe the behaviour of a zero or two-fold positive eigenvalue $\lambda(x, y)$ near the stability boundary. Let us consider the function $\lambda(x, y)$ as a one-parameter family of characteristic curves $\lambda(x)$, where y is the parameter of the family. Equations (4.2) and (4.3) enable us to investigate the behaviour of the characteristic curve $\lambda(x)$, qualitatively and quantitatively, in the vicinity of the stability boundary.

If the stability domain (S) borders on the divergence domain (D), then $\lambda(x, y)$ is locally a family of real quadratic parabolas $\lambda(x)$, described by Eq. (4.2). The orientation of the parabola in the (Re λ, x) plane is determined by the sign of P. As follows from relations (4.1), this quantity is also responsible for the convexity of the section of the stability domain by the plane of the parameters x and y. When $P < 0$, the parabola $\lambda(x)$ is convex downward, and the section of the stability domain is concave (Fig. 2). Figure 2 shows the evolution of the characteristic curve as y varies from positive to negative values. For $y > 0$ all points of the characteristic curve lie above the x axis, guaranteeing the stability of the system. At $y = 0$ the characteristic curve $\lambda(x)$ is tangent to the abscissa axis at the point $x = 0$, implying the formation of a zero eigenvalue at a regular point $\mathbf{p}_0$ of the boundary between the stable and divergence domains. When $y < 0$, because of the concavity of the section of the stability domain, an interval of the parameter x appears

$$x^2 < -y|\mathbf{f}|/|P| \tag{4.9}$$

in which the eigenvalue becomes negative (divergence). Comparing inequalities (4.1) and (4.9), we see that the latter is also a quadratic approximation to the divergence domain. In the case when **P** > 0, the parabola $\lambda(x)$ is convex upward, the section of the stability domain is convex, and the pattern of the behaviour of the characteristic curve as a function of the parameter y is obtained from the "frames" shown in Fig. 2 by reflection in a horizontal axis passing through the apex of the parabola.

The behaviour of the characteristic curves $\lambda(x)$ near the boundary between the stability (S) and flutter (F) domains is more complicated, since in that case the curves may be modified. The type of metamorphosis depends on whether the section of the stability domain by the plane through the normal and tangent vectors to the boundary of that domain is convex or concave. It follows from the second inequality of (4.1) that the convexity of the section is determined here by the sign of Q.

When $Q < 0$, the section of the stability domain is concave and the flutter domain is convex (Fig. 3). Equation (4.3) describes a family of three-dimensional characteristic curves $\lambda(x)$ whose evolution as a function of the parameter y is shown in Fig. 3. When $y > 0$, the characteristic curves are the two branches of a hyperbola

$$(\mathrm{Re}\,\lambda - \lambda_0 - Rx)^2 + Qx^2 = y|\mathbf{f}|, \quad \mathrm{Im}\,\lambda = 0 \tag{4.10}$$

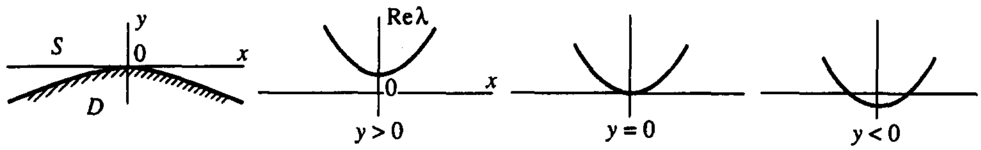

Fig. 2

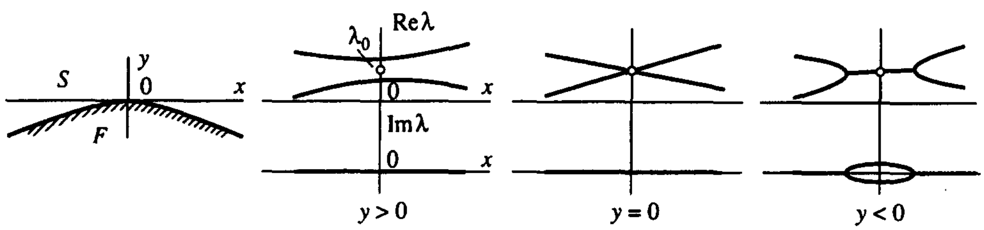

Fig. 3

in the $(\mathrm{Re}\lambda, x)$ plane. As the parameter y decreases, the curves $\lambda(x)$, moving along the real axis, approach one another, having a common point $(\lambda_0, 0)$ at $y = 0$ (Fig. 3). In the neighbourhood of the common point, the characteristic curves are linear functions of x

$$\mathrm{Re}\,\lambda = \lambda_0 + x(R \pm \sqrt{-Q}),\ \mathrm{Im}\,\lambda = 0 \tag{4.11}$$

When y is reduced further, one has a qualitative change in the behaviour of the characteristic curves: an interval of the parameter x appears

$$x^2 < -y|\mathbf{f}|/|Q| \tag{4.12}$$

in which the curves $\lambda(x)$ leave the real plane, forming an ellipse of complex eigenvalues – an "instability bubble" [14]

$$\mathrm{Re}\lambda = \lambda_0 + xR,\ (\mathrm{Im}\lambda)^2 - x^2Q = -y|\mathbf{f}| \tag{4.13}$$

Outside the interval (4.12) the characteristic curves are the two branches of the adjacent hyperbola (4.10) lying in the real plane. Comparison of formulae (4.10) and (4.13) indicates that the ellipse and the hyperbola lie in orthogonal planes, touching one another at the points $x_{1,2} = \pm\sqrt{y|\mathbf{f}|/Q}$, where two-fold real eigenvalues are formed

$$\lambda_{1,2} = \lambda_0 \pm R\sqrt{y|\mathbf{f}|/Q} \tag{4.14}$$

Thus, as the parameter x varies, the eigenvalue bifurcates at the points x_1 and x_2: two eigenvalues, being in one plane, collide and "take off" in directions orthogonal to that plane. This behaviour of the eigenvalues, known as "strong interaction," is typical of transverse crossing of the boundary of the flutter domain [9, 14]. Note that inequality (4.12), which defines the boundaries of the "instability bubble," is identical with the quadratic approximation to the section of the flutter domain. Thus the phenomenon of overlapping characteristic curves turns out to be closely related to the convexity of the flutter domain.

When $Q > 0$, the section of the stability domain is convex and the flutter domain is concave. The corresponding metamorphosis of characteristic curves is shown in Fig. 4. When $y > 0$, the eigenvalues form a real ellipse (4.10) in the interval

$$x^2 < y|\mathbf{f}|/Q \tag{4.15}$$

and a hyperbola (4.13) lying in the $(\mathrm{Re}\lambda, \mathrm{Im}\lambda, x)$ space. As the parameter y is reduced, the branches of the hyperbola approach one another and the "stability bubble" shrinks (Fig. 4). At $y = 0$ the ellipse (4.10) becomes a point $(\lambda_0, 0)$, in whose neighbourhood the characteristic curves are linear functions of x

$$\mathrm{Re}\,\lambda = \lambda_0 + Rx,\ \mathrm{Im}\,\lambda = \pm x\sqrt{Q} \tag{4.16}$$

Hyperbolas (4.13), consisting of complex-conjugate eigenvalues, correspond to negative values of y (Fig. 4).

Remark. In some cases it is preferable to work with frequency curves $\omega(x)$, rather than with characteristic curves $\lambda(x)$. Functions describing the metamorphosis of the frequency curves locally are obtained from Eqs (4.2) and (4.3) by the substitution $\lambda = \omega^2$. When that is done it is readily seen that, near the boundary between the stability and flutter domains, the type of metamorphosis undergoes no qualitative changes. Near the boundary between stability and divergence domains, however, the situation is different. At $x = 0, y = 0$ we have, corresponding to a simple

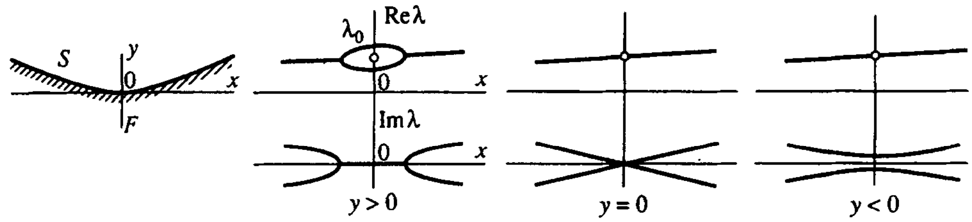

Fig. 4

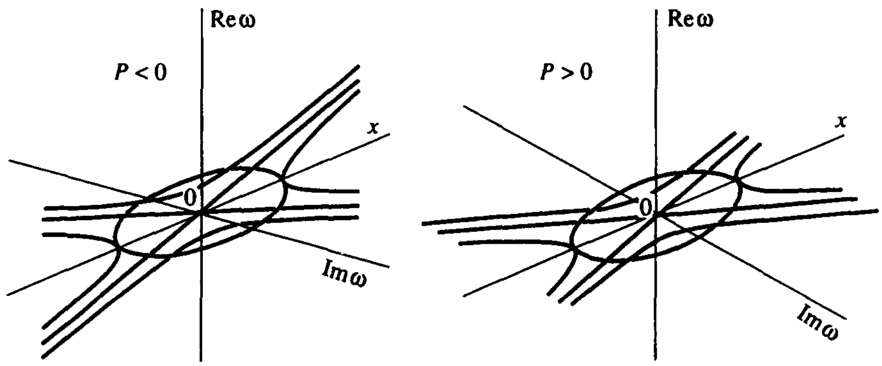

Fig. 5

eigenvalue $\lambda_0 = 0$, a two-fold zero frequency with Jordan chain of length 2 [9]. Formula (4.2) with $\lambda = \omega^2$ describes the bifurcation of this frequency as a function of the parameters x and y. The pattern of metamorphosis of frequency curves $\omega(x)$ as a function of the parameter y resembles the behaviour of the characteristic curves $\lambda(x)$ near the boundary of the flutter domain. The differences is that the frequency curves may lie only in the real and imaginary planes (Fig. 5). Thus, metamorphosis of frequency curves near the boundary of the divergence domain, as described by Eq. (4.2) with $\lambda = \omega^2$ (Fig. 5), is qualitatively different from the behaviour of the characteristic curve $\lambda(x)$ as described by Eq. (4.2) (Fig. 2).

5. EXAMPLE. THE STABILITY OF THE VIBRATIONS OF A PLATE IN A GAS FLOW

As an example, let us consider a model problem on the stability of a plate in a gas flow. The plate rests on two elastic supports with stiffnesses c_1 and c_2 per unit span and has two degrees of freedom: a vertical displacement z and an angle of deviation φ (Fig. 6). Small vibrations of the plate are described, in dimensionless variables, by the following equations [10, 15]

$$\left\| \begin{matrix} \ddot{z} \\ \ddot{\varphi} \end{matrix} \right\| + \mathbf{A} \left\| \begin{matrix} z \\ \varphi \end{matrix} \right\| = 0, \quad \mathbf{A} = \left\| \begin{matrix} 1 & c-q \\ 12c & 3-3q \end{matrix} \right\|; \quad q = \frac{c_y^\alpha \rho v^2}{2(c_1+c_2)}, \quad c = \frac{c_1 - c_2}{2(c_1+c_2)} \tag{5.1}$$

where q is the load parameter, which is proportional to the velocity head, c_y^α is the coefficient of lift force, ρ and v are the gas density and flow velocity respectively, and c is a parameter characterizing the relation between the stiffnesses of the supports. It is assumed that the point Y at which the lift force is applied lies at a distance equal to a quarter of the plate's width from its front edge. Thus, system (5.1) depends on the parameter vector $\mathbf{p} = (c, q)$. Setting $c_y^\alpha > 0$, we obtain $q \geqslant 0$. In addition, it follows from physical considerations that $-1/2 \leqslant c \leqslant 1/2$.

Seeking a solution in the form $[z, \varphi]^T = \mathbf{u}e^{i\omega\tau}$, we arrive at the eigenvalue problem

$$\mathbf{Au} = \lambda\mathbf{u}, \ \lambda = \omega^2$$

The corresponding characteristic equation

$$\lambda^2 + (3q-4)\lambda + 12cq - 3q - 12c^2 + 3 = 0 \tag{5.2}$$

yields equations of curves separating the plane of the parameters c and q into domains of stability, divergence and flutter (Fig. 6)

$$q_f(c) = \frac{2}{3}(1 + 4c \pm 2\sqrt{c(c+2)}), \quad q_d(c) = \frac{1-4c^2}{1-4c} \tag{5.3}$$

The curve $q_f(c)$ bounds the flutter domain, and part of the curve $q_d(c)$, up to the point $J(c = 2/3 - \sqrt{13}/6, q = 4/3)$ at which these curves are tangent to one another, is the boundary between the stability and divergence domains. The dashed line in Fig. 6 represents the part of the curve $q_d(c)$ lying in the divergence domain.

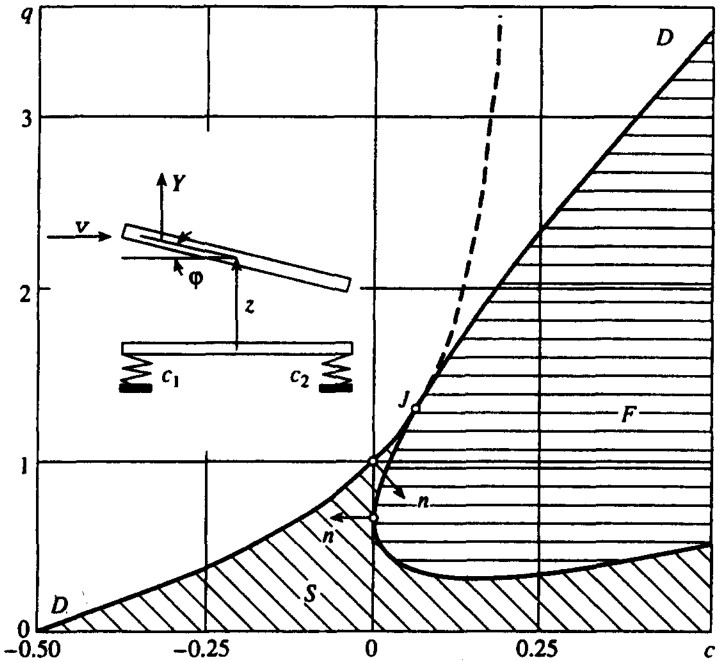

Fig. 6

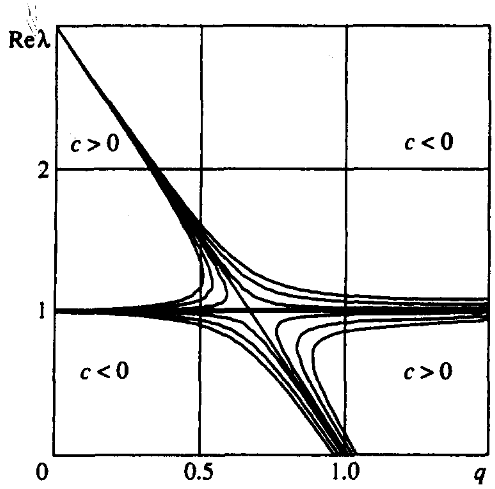

Fig. 7

Let us consider the point ($c = 0$, $q = 2/3$) on the boundary between the stable and flutter domains corresponding to the two-fold eigenvalue $\lambda = 1$ (Fig. 6). Characteristic equation (5.2) becomes

$$\left(\lambda - 1 + \frac{3}{2}q - 1\right)^2 - \left(-4c + \frac{3}{2}q - 1\right)^2 = -8c - 4c^2 \tag{5.4}$$

At $c = 0$ Eq. (5.4) defines two straight lines

$$\lambda = 1, \quad \lambda = 3 - 3q \tag{5.5}$$

intersecting at the point ($q = 2/3$, $\lambda = 1$). If $c \neq 0$, Eq. (5.4) describes a family of hyperbolas with asymptotes (5.5) (Fig. 7). For small $c < 0$ and $0 \leqslant q \leqslant 1$, the solutions $\lambda(q)$ of Eq. (5.4) lie in the real plane, one of the eigenvalues remaining positive throughout the interval in which q varies, while the second changes sign at some $q_d < 1$ (Fig. 7). Consequently, for $c < 0$ and sufficiently large q, system

(5.1) becomes statically unstable (divergence). The change of sign in the parameter c entails passage to hyperbolas laying in the adjacent angles formed by asymptotes (5.5). The characteristic curves are modified, and at the same time a zone of complex eigenvalues appears and a sudden fall in the critical load occurs. At the same time, stability is lost at those values of q_f where the two positive eigenvalues λ come together to form a complex-conjugate pair (flutter).

We will show that Eq. (5.4), which describes the metamorphosis of the characteristic curves $\lambda(q)$, may be approximated by formula (1.5), whose coefficients are found based solely on information about the system at the point $\mathbf{p}_0 = (0, 2/3)$.

In fact, $\lambda(2/3) = 1$ is a two-fold eigenvalue whose eigenventors and associated vectors are

$$\mathbf{u}_0 = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix}, \quad \mathbf{v}_0 = \begin{Vmatrix} 0 \\ -2/3 \end{Vmatrix}, \quad \mathbf{u}_1 = \begin{Vmatrix} 0 \\ -3/2 \end{Vmatrix}, \quad \mathbf{v}_1 = \begin{Vmatrix} 1 \\ 0 \end{Vmatrix} \tag{5.6}$$

Since at the point $(0, 2/3)$ we have

$$\mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial c} \mathbf{u}_0 = -8, \quad \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_0 = 0 \tag{5.7}$$

it follows that condition (1.3) is satisfied. A simple calculation shows that

$$\mathbf{v}_1^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_0 = 0, \quad \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_1 = -3, \quad \mathbf{v}_0^T \frac{\partial^2 \mathbf{A}}{\partial q^2} \mathbf{u}_0 = 0, \quad \mathbf{v}_0^T \frac{\partial \mathbf{A}}{\partial q} \mathbf{G}_0 \frac{\partial \mathbf{A}}{\partial q} \mathbf{u}_0 = 0 \tag{5.8}$$

Substituting expressions (5.7) and (5.8) into Eq. (1.5) and taking into account that $\Delta p_1 = c$, $\Delta p_2 = q - 2/3$, we obtain the required approximation

$$\left(\lambda - 1 + \frac{3}{2} q - 1\right)^2 - \left(\frac{3}{2} q - 1\right)^2 = -8c \tag{5.9}$$

Comparing the exact equation (5.4) with the approximate one (5.9), we note that the asymptotes $\lambda = 1$ and $\lambda = 3 - 3q$ coincide and that the characteristic curves are well approximated at small values of c. A quadratic approximation of the flutter domain in the neighbourhood of the point $(0, 2/3)$ is found from the condition that the discriminant of Eq. (5.9) must be positive

$$c > \frac{9}{32}\left(q - \frac{2}{3}\right)^2$$

The approximation (5.9) corresponds to a convex flutter domain (Fig. 6), while the approximate equation of the flutter boundary in the neighbourhood of the point $\mathbf{p}_0 = (0, 2/3)$ is in good agreement with the exact equation of the boundary $q_f(c)$ in (5.3).

We now consider a point $(c = 0, q = 1)$ on the boundary between the stability and divergence domains, where the matrix $\mathbf{A}$ of system (5.1) has simple eigenvalues $\lambda = 1$ and $\lambda = 0$. Let us calculate the normal vector $\mathbf{f}$ to the boundary of the divergence domain. We first find the eigenvectors of the zero eigenvalue

$$\mathbf{u}_0 = \begin{Vmatrix} 1 \\ 1 \end{Vmatrix}, \quad \mathbf{v}_0 = \begin{Vmatrix} 0 \\ 1 \end{Vmatrix}$$

Substituting these expressions into formulae (3.1), we obtain

$$\mathbf{f} = \begin{Vmatrix} 12 \\ -3 \end{Vmatrix}, \quad |\mathbf{f}| = 3\sqrt{17}$$

The orthogonality condition (3.7) enables us to find the unit tangent vector $\mathbf{t}$, and formula (3.2) yields the matrix $\mathbf{H}$

$$\mathbf{t} = \frac{1}{\sqrt{17}} \begin{Vmatrix} 1 \\ 4 \end{Vmatrix}, \quad \mathbf{H} = \begin{Vmatrix} -132 & 12 \\ 12 & 0 \end{Vmatrix}$$

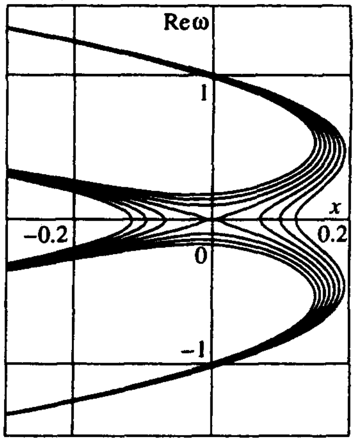

Fig. 8

Substituting these expressions into Eq. (4.1), we obtain an approximate equation for the boundary between the stability and divergence domains in x, y coordinates

$$y = -\frac{12\sqrt{17}}{289}x^2$$

Since the y coordinate increases in the direction of the normal vector **f**, the divergence domain is convex at the point (0, 1). This agrees with the conclusions of Section 3 concerning the convexity of the divergence domain for a negative-definite matrix **H**. The metamorphosis of the frequency curve $\omega(x)$ in the neighbourhood of that point is approximated by Eq. (4.2) with $\lambda = \omega^2$, which takes the form

$$\omega^2 - \frac{36}{17}x^2 = 3\sqrt{17}y \tag{5.10}$$

To get an idea of the accuracy of approximation (5.10), let us compare it with the characteristic equation (5.2) written in the system of coordinates defined by the vectors **t** and $\mathbf{n} = \mathbf{f}/|\mathbf{f}|$. After substituting the parameters

$$c = \frac{1}{\sqrt{17}}(x + 4y), \quad q = 1 + \frac{1}{\sqrt{17}}(4x - y)$$

the characteristic equation (5.2) becomes

$$\omega^2 - \frac{36}{17}x^2 - 3\sqrt{17}y = \omega^4 + \omega^2\frac{3}{\sqrt{17}}(4x - y) + \frac{12}{17}y(7x - 20y) \tag{5.11}$$

Comparison of the exact equation (5.11) with the approximate one (5.10) shows that the latter, being far simpler, is a good description of the metamorphosis of the frequency curves in the vicinity of the point ($x = 0$, $\omega = 0$) for small y values (Fig. 8).

This research was supported by the International Association for the Promotion of Cooperation with Scientists from the Independent States of the Former Soviet Union (INTAS YSF 01/1-057).

REFERENCES

1. CLAUDON, J.-L., Characteristic curves and the optimum design of two structures subjected to circulatory loads. *J. Mec.*, 1975, **14**, 3, 531–543.
2. HANAOKA, M. and WASHIZU, K., Optimum design of Beck's column. *Computers and Structures*, 1980, **11**, 6, 473–480.
3. KOUNADIS, A. N. and KATSIKADELIS, J. T., On the discontinuity of the flutter load for various types of cantilevers. *Int. J. Solid Structures*, 1980, **16**, 4, 375–383.
4. SEYRANIAN, A. P. and SHARANYUK, A. V., Sensitivity and optimization of critical parameters in dynamic stability problems. *Izv. Akad. Nauk SSSR. MTT*, 1983, 5, 174–183.

5. LANGTHJEM, A. P. and SUGIYAMA, Y., Optimum shape design against flutter of a cantilevered column with an end-mass of finite size subjected to a non-conservative load. *J. Sound Vibrations*, 1999, **226**, 1, 1–23.
6. BOLOTIN, V. V., *Non-conservative Problems of the Theory of Elastic Stability*. Pergamon Press, Oxford, 1963.
7. ZIEGLER, H., *Principles of Structural Stability*. Blaisdell, Waltham, MA, 1968.
8. ARNOLD, V. I., *Geometrical Methods in the Theory of Ordinary Differential Equations*. Springer, New York and Berlin, 1983.
9. SEYRANIAN, A. P., Bifurcations in one-parameter circulatory system. *Izv. Ross. Akad. Nauk. MTT*, 1994, 1, 142–148.
10. SEYRANIAN, A. P. and KIRILLOV, O. N., Bifurcation diagrams and stability boundaries of circulatory systems. *Theor. Appl. Mech.*, 2001, 26, 135–168.
11. VISHIK, M. I. and LYUSTERNIK, L. A., Solution of some perturbation problems in the case of matrices and selfadjoint and non-selfadjoint differential equations. *I. Uspekhi Mat. Nauk*, 1960, **15**, 3, 3–80.
12. YAKUBOVICH, V. A. and STARZHINSKII, V. M., *Parametric Resonance in Linear Systems*. Nauka, Moscow, 1987.
13. SEYRANIAH, A. P. and MAILYBAEV, A. A., On stability boundaries of conservative system. *ZAMP*, 2001, **52**, 4, 669–679.
14. SEYRANIAN, A. P., Sensitivity analysis of eigenvalues and the development of instability. *Stroinicky Casopis*, 1991, **42**, 3, 193–208.
15. PANOVKO, Ya. G. and GUBANOVA, I. I., *Stability and Oscillations of Elastic Systems*. Consultants Bureau, New York, 1965.

Translated by D.L.