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Bifurcation diagrams and stability boundaries of circulatory systems

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Abstract

Stability problems of linear circulatory systems of general type with finite degrees of freedom depending on two parameters are considered. It is shown that these systems in the generic case are subjected to flutter and divergence instabilities. Bifurcations of eigenvalues describing mechanism of static and dynamic losses of stability are studied, and geometric interpretation of these catastrophes is given. For two-dimensional case boundaries between stability, flutter and divergence domain and generic singularities of these boundaries are analyzed. With the use of the left and right eigenvectors and associated vectors tangent cones and normal vectors to the boundaries are calculated. As an example stability of a rigid panel vibrating in airflow is considered and discussed in detail.

1 Introduction

The equation of motion of a linear autonomous system with nonconservative positional forces is

$$M\ddot{q} + Cq = 0, \tag{1}$$

where M is a symmetric positive definite $m \times m$ matrix, C is a non-symmetric matrix of the same order, and q is a vector of generalized coordinates of dimension m . Dots mean differentiation with respect to time t. System (1) is usually called a circulatory system, see [20]. Using the transformation $q = ue^{\nu t}$ we obtain an eigenvalue problem

$$Cu + \nu^2 Mu = 0, \tag{2}$$

u being an eigenvector. Characteristic exponents ν are determined from the characteristic equation

$$\det\left[C + \nu^2 M\right] = 0. \tag{3}$$

Since C and M are real matrices, it follows that if any ν satisfies (3), then $-\nu$, $\bar{\nu}$, and $-\bar{\nu}$ also satisfy this equation. It means that system (1) can be stable if and only if all the characteristic exponents belong to the imaginary axis of the complex plane. If at least one of ν is real while remaining characteristic exponents belong to the imaginary axis, then system (1) is statically unstable (divergence). And if at least one of the characteristic exponents ν is complex this is termed kinetic instability (flutter). Introducing the notation

$$A = M^{-1}C, \quad \lambda = -\nu^2, \tag{4}$$

we obtain an eigenvalue problem for the matrix operator

$$Au = \lambda u \tag{5}$$

It is easy to see that in terms of λ system (1) is stable if all λ are positive and simple eigenvalues; if all eigenvalues λ are real and some of them negative system (1) is statically unstable (divergence). And if at least one eigenvalue λ is complex it means flutter instability.

Stability problems for circulatory systems have been considered in [2], [13], [4], [20], [5], [7], [18], [8]. In these works many specific problems, dependent on one and two parameters, have been studied.

The aim of the present paper is to study stability, flutter and divergence boundaries for two-parameter circulatory systems in the generic case, i.e. typical case when singularities do not disappear with a change of the family of matrices, see [1]. Stability analysis done in this paper is based on bifurcation diagrams of matrices, see [1] and [3], and perturbation theory of eigenvalues and eigenvectors by [19] and [10], see also [12]. This theory has been developed and used by [16], [15], [6], and [11] to study stability problems of mechanical systems.

First we consider the case of dependence of circulatory systems on a single parameter p. This case has been studied by [17]. For the sake of convenience we begin with a short outline of that paper.

For families of matrices A smoothly depending on a single parameter p it is known, see [1] and [3], that the generic case is characterized by simple eigenvalues λ, and only at some isolated points p_0 by double real eigenvalues λ_0 with Jordan chain of second order. More complicated singularities of $A(p)$ can be destroyed by infinitesimal shift of the family.

It is easy to show that p_0 is a boundary of the stability domain. Indeed, if we take an increment $p = p_0 + \Delta p$ we obtain the increment of the matrix A in the form

$$A = A_0 + A_1 \Delta p + ..., \tag{6}$$

where

$$A_0 = A(p_0), \quad A_1 = \left(\frac{dA}{dp} \right)_{p=p_0}. \tag{7}$$

These matrices are connected with the matrices and by the relations

$$A_0 = M_0^{-1} C_0, \quad A_1 = M_0^{-1} C_1 - M_0^{-1} M_1 M_0^{-1} C_0,$$

$$M_0 = M(p_0), \quad C_0 = C(p_0), \tag{8}$$

$$M_1 = \left(\frac{dM}{dp} \right)_{p=p_0}, \quad C_1 = \left(\frac{dC}{dp} \right)_{p=p_0}$$

Considering λ_0 as a double real eigenvalue with Jordan chain of second order with the right and left eigenvectors u_0 and v_0 we find bifurcation of λ_0 in the form, see (A17, A21)

$$\lambda = \lambda_0 \pm \sqrt{f_1 \Delta p} + O\left(|\Delta p|\right) \tag{9}$$

$$f_1 = (A_1 u_0, v_0). \tag{10}$$

Since $(A_1 u_0, v_0)$ is real, the quantity $f_2 = 0$, while f_1 is real and doesn't depend on the increment Δp.

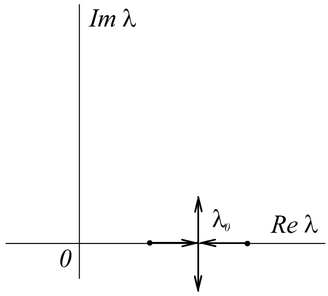

Fig. 1. Strong interaction of two eigenvalues

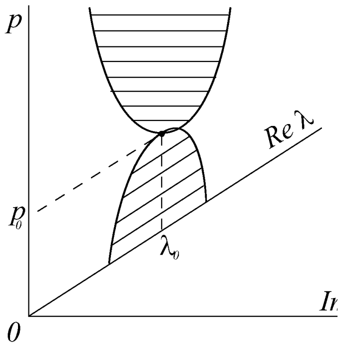

Fig.2. Bifurcation of eigenvalues shown in three-dimensional space

The bifurcation (9) is illustrated in Fig.1 and Fig.2. It can be interpreted as *strong interaction* of two eigenvalues, see [16]. If $f_1 < 0$, then with an increase of $p < p_0$ two real eigenvalues come together, merge at $p = p_0$, and then diverge along a straight line parallel to the imaginary axis. The arrows in Fig.1 show the direction of motion of λ when p increases. If $f_1 > 0$, the direction of motion of eigenvalues changes to the opposite. The bifurcation (9) is shown in Fig.2 in three-dimensional space ($f_1 < 0$). At $p \approx p_0$ the intersecting curves are quadratic parabolas of the same curvature lying in the orthogonal planes $\mathrm{Im}\,\lambda = 0$ and $\mathrm{Re}\,\lambda = \lambda_0$.

Since the matrix A is real, the interaction pictures in Fig.1 and Fig.2 are symmetric with respect to axis and plane $\mathrm{Im}\,\lambda = 0$, respectively. It is easy to see that if λ is a simple and real eigenvalue it remains real with a change of p . Indeed, in the other case the complex conjugate $\bar{\lambda}$ also appears, which means an increase of the total number of roots of the characteristic equation. Hence, with a change of p eigenvalues are able to leave the real axis (or the plane $\operatorname{Im}\lambda = 0$) only when they meet and interact strongly according to (9), see Fig.1 and Fig.2.

Let us return to problem (1), (2). Since $\lambda = -\nu^2$ positive eigenvalues λ correspond to pure imaginary characteristic exponents $\nu = \pm i\sqrt{\lambda}$, and negative λ correspond to real $\nu = \pm i\sqrt{|\lambda|}$. Thus, positive $\lambda_0 = -\nu_0^2$ at the bifurcation point (9) means transition of stability of system (1) to kinetic instability (flutter) or vice versa (depending on the sign of f_1), and negative λ_0 corresponds to transition of kinetic instability (flutter) to aperiodic instability (divergence) or vice versa.

The bifurcation (9) can be expressed through characteristic exponents ν. Substituting $p - p_0$ instead of Δp and using $\lambda_0 = -\nu_0^2$ we obtain

$$\nu = \pm\nu_0 \left(1 \pm \frac{\sqrt{f_1(p - p_0)}}{2\nu_0^2} \right) + O\left(|\Delta p|\right) \tag{11}$$

The interaction (11) is illustrated in Fig.3 a,b assuming $f_1 < 0$. Direction of the arrows corresponds to increasing p. If $f_1 > 0$, then the arrows should be reversed.

Besides catastrophes (9), (11) circulatory system (1) can lose stability statically which corresponds to passing of positive eigenvalues λ through zero point to negative values (divergence). Expansion of a simple eigenvalue in the vicinity of $\lambda_0 = 0$ leads to the relation, see (A4), (A9), (A11)

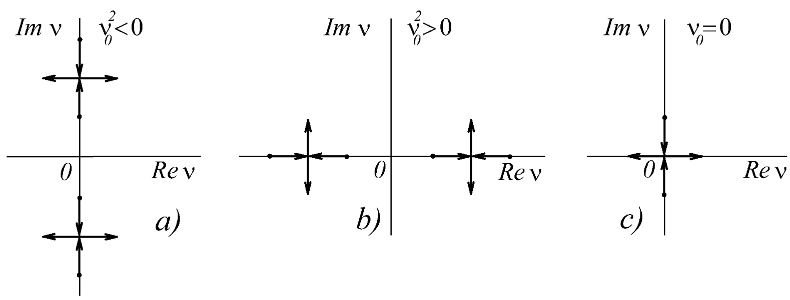

Fig. 3. Three types of generic catastrophes for one-parameter circulatory systems

$$\lambda = g_1 \Delta p + o\left(|\Delta p|\right), \tag{12}$$

$$g_1 = (A_1 u_0, v_0). \tag{13}$$

In (13) u_0 and v_0 are the eigenvectors corresponding to $\lambda_0 = 0$. In terms of characteristic multipliers we have

$$\nu = \pm\sqrt{g_1(p - p_0)} + O\left(|\Delta p|\right) \tag{14}$$

For $g_1 < 0$ the behavior of ν with the increase of p is shown in Fig.3c. If $g_1 > 0$ direction of motion of ν changes to the opposite.

The main result for stability of one-parameter circulatory systems is formulated as follows, see Seyranian (1994).

Theorem 1 *One-parameter circulatory systems in the generic case are subjected to catastrophes of three types: flutter, transition of flutter to divergence, and divergence.*

These catastrophes are described by relations (11), (14) and are shown in Fig.3. Three-dimensional pictures of eigenvalue interaction are similar to Fig.2.

Many authors have considered one-parameter circulatory systems. In the book by Leipholz (1987) behavior of eigenvalues of matrices depending on a single parameter has been studied. Fig.4 reproduces Fig.6c,d in [8], showing the behavior of eigenvalues λ of a nonsymmetric matrix depending on a parameter p with p_c denoting the critical value of the parameter p. We note that such behavior of λ contradicts the presented results, and is therefore impossible.

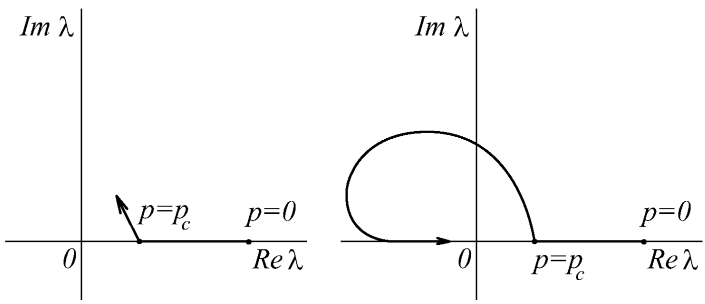

Fig. 4. Behavior of eigenvalues depending on one parameter [8]

In many books and papers a typical plot of dependence of p on λ is given, which is reproduced in Fig.5, and as a flutter condition the equality $(dp/d\lambda) = 0$ is used, see for example [14]. We note that at $p > p_c$ the eigenvalues don't disappear, and double eigenvalues λ at the critical point p_c are not differentiable with respect to p.

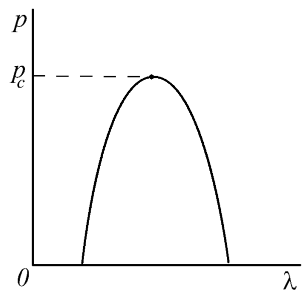

Fig. 5. Typical plot of flutter instability [14]

2 Two-parameter circulatory systems

We consider circulatory system (1) assuming that the matrices M and C smoothly depend on two parameters p_1, p_2 and would like to study boundaries between stability, flutter, and divergence domains on the parameter plane. For appropriate eigenvalue problem we use equation (5).

According to [3] and [1] bifurcation diagram of a general two-parameter family of real matrices A in the plane of parameters has the form of isolated point and a plane curve, whose only singularities are cusps and nodes, Fig.6. Here Jordan forms of matrices A are denoted by product of determinants of their blocks, for example α^2 means Jordan block of second order with the eigenvalue α etc. The isolated point in Fig. 6. corresponds to Jordan block of second order with double complex eigenvalue $\xi + i\zeta$ (i.e. a double complex eigenvalue with a single eigenvector). The cusps on the curve correspond to matrices having 3×3 Jordan block with 3-fold real eigenvalue α, and the nodes correspond to matrices containing two Jordan blocks with two 2×2 different real eigenvalues α and β. Other points on the curve correspond to matrices having single 2×2 Jordan block with double real eigenvalue α, and points off the curve to matrices with distinct eigenvalues. If the family possesses matrices of more complicated type, or if bifurcation diagram has worse singularities, then they can be destroyed by an arbitrary small shift in the family of matrices, see [1].

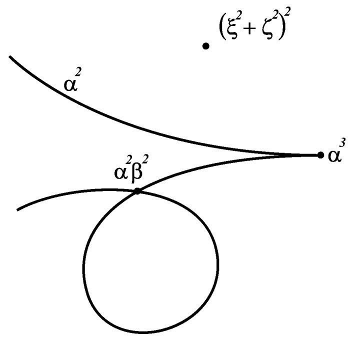

Fig. 6. Two-parameter bifurcation diagram [3]

Let us consider a point p_0 on a curve, corresponding to a double eigenvalue α_0, Fig.6. We take an increment of the vector of parameters in the form $p = p_0 + \varepsilon e + \varepsilon^2 d$, where $e = (e_1, e_2)$ and $d(d_1, d_2)$ are arbitrary vectors of the variation, and $\varepsilon > 0$ is a small number. This can be interpreted as emitting of smooth curves $p(\varepsilon)$ in the vicinity of the point p_0. In the sequel we take the vector $d = 0$, i.e. we consider linear variations $p = p_0 + \varepsilon e$, unless otherwise stipulated. Then, according to (A17), (A21) we obtain

$$\lambda = \alpha_0 \pm \sqrt{\langle f_1, e \rangle \, \varepsilon} + O\,(\varepsilon)\,. \tag{15}$$

In this expression the real vector f_1 in accordance with (A22) is determined by the right and left eigenvectors corresponding to α_0. As α_0 is a real number these vectors are also real. Hence, due to (A22) we should take $f_2 = 0$. If the expression under square root in (15) is positive, then the double eigenvalue α_0 splits into two real and simple λ. And if this expression is negative then α_0 splits into two complex-conjugate eigenvalues .

For the following presentation we need to introduce a concept of *tangent cone*, see [9]. Tangent cone to a set Z at its boundary point is a set of direction vectors of the curves starting at this point and lying in the set Z. A tangent cone is *nondegenerate*, if it cuts out on a sphere a set of nonzero measure. Other-wise, the cone is called *degenerate*.

A) First we consider a point α_0 at the curve α^2 on the bifurcation diagram (Fig.6), assuming that $\alpha > 0$ and remaining eigenvalues λ are simple and positive. Then from (15) it follows that for rather small ε if $\langle f_1, e\rangle > 0$, then the vectors ε_e lie in the stability domain, and if $\langle f_1, e\rangle < 0$, then the vectors ε_e belong to the flutter domain. The tangent cone to the stability domain is given by

$$K_s = \left\{e \in R^2 : \langle f_1, e\rangle > 0\right\}.$$

Thus, the curve $\alpha^2(\alpha > 0)$ is a boundary between stability and flutter domains, and the vector f_1 is the normal vector to this boundary lying in the stability domain, see Fig.7a.

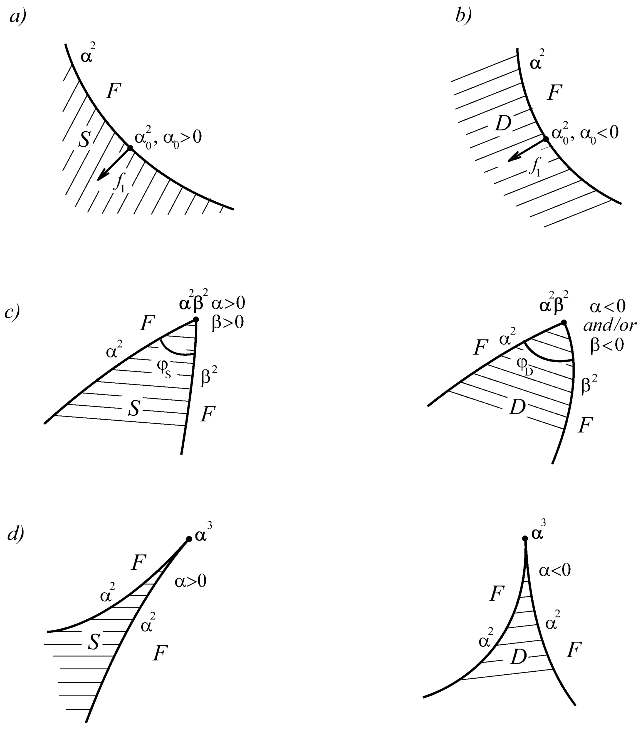

Fig 7.(a-d) List of generic singularities appearing at the stability, flutter, and divergence boundaries for two-parameter circulatory systems

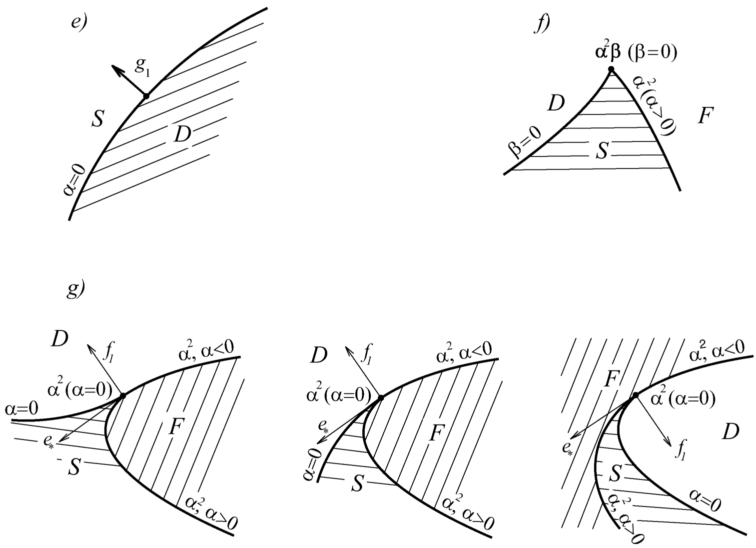

Fig 7.(e-g)

Let us study convexity properties of the boundary. For this purpose we take a vector of variation e_* tangent to the boundary, i.e. $\langle f_1, e_* \rangle = 0$. In this case according to (A23), (A27) a double eigenvalue f_1 splits into two eigenvalues $\lambda = \alpha_0 + \varepsilon\lambda_2 + o(\varepsilon)$., where the coefficients λ_2 are determined from the quadratic equation (A27). Since α_0 is real number, the imaginary parts of the coefficients a_1, a_2 of (A27) are zero. Then according to (A29) we find

$$a_1 = \langle b_1, e_* \rangle, \quad a_2 = \langle H_1 e_*, e_* \rangle, \tag{16}$$

where b_1, H_1 are respectively the real vector and the square matrix of dimension $n = 2$, defined in (A28) via left and right eigenvectors and associated vectors.

Having solved (A27) with the use of (16) we obtain expressions for λ up to the terms of $o(\varepsilon)$

$$\lambda = \alpha_0 - \frac{\langle b_1, e_* \rangle \pm \sqrt{D}}{2}\varepsilon$$

$$D = \langle b_1, e_* \rangle^2 - 4 \langle H_1 e_*, e_* \rangle \tag{17}$$

Since the discriminant D is a quadratic form on the components of the vector e_* the quantity D is independent on sign of e_*. If $D(e_*) < 0$, then from (17) it follows that the vectors e_* and -e_* lie in flutter domain because these vectors correspond to complex-conjugate λ. If $D(e_*) > 0$, then double $\alpha_0 > 0$ splits into two real λ. Consequently, the vectors e_* and -e_* belong to the stability domain.

The case $D(e_*) = 0$ needs further investigation. It might mean nonsplitting λ or splitting of λ in higher order of ε.

Thus, if $D(e_*) > 0$ at the point $p = p_0$, then flutter domain is convex at this point. And the condition $D(e_*) < 0$ means concavity of flutter domain. Hence, knowing at $p = p_0$ double eigenvalue $\alpha_0 > 0$ and corresponding eigenvectors and associated vectors allows to construct a normal vector to the boundary between stability and flutter domains on the plane p_1, p_2 and reveal what of halfplanes with respect to the tangent vector does the flutter domain belong to, and is it convex or concave.

B) Consider now a point α_0 on the curve α^2 assuming that $\alpha < 0$ and remaining λ are simple and real. With the use of (15) we find that the cone $(f_1, e) > 0$ lies in divergence domain, and the cone $(f_1, e) < 0$ corresponds to flutter domain. Thus, the curve α^2 $(\alpha < 0)$ is a boundary between flutter and divergence domains, and the vector f_1 is a normal vector to this boundary lying in the divergence domain, see Fig.7b. Convexity or concavity of flutter domain can be established with the aid of (17). If at $p = p_0$ the discriminant $D(e_*) > 0$, then flutter domain is convex, and if $D(e_*) < 0$ flutter domain is concave at this point.

C) Consider on the bifurcation diagram the intersection point $\alpha^2\beta^2$, Fig.7c. This point corresponds to two Jordan blocks of second order with different real eigenvalues α and β. It is easy to see that this singularity means an angular point at the boundary between stability and flutter, or divergence and flutter domains. Let us calculate a tangent cone to the stability domain assuming $\alpha > 0$, $\beta > 0$. According to (15) splitting of double eigenvalues α and β is given by the relations

$$\lambda = \alpha \pm \sqrt{\langle f_1^{\alpha}, e\rangle \varepsilon} + O(\varepsilon)$$

$$\lambda = \beta \pm \sqrt{\left\langle f_1^{\beta}, e\right\rangle \varepsilon} + O(\varepsilon), \qquad (18)$$

where the vectors $f_1^{\alpha}, f_1^{\beta}$ correspond to the eigenvalues α and β. From (18) we find that the tangent cone to the stability domain is described by the expression

$$K_s = \left\{ e \in R^2 : \langle f_1^{\alpha}, e \rangle > 0, \quad \left\langle f_1^{\beta}, e \right\rangle \right\} > 0 \tag{19}$$

According to (18) all the direction vectors e, satisfying the inequalities $\langle f_1^{\alpha}, e \rangle < 0$ or $\left\langle f_1^{\beta}, e \right\rangle < 0$, correspond to splitting of α (or β) to complex-conjugate quantities which means flutter instability. Therefore, at the point $\alpha^2\beta^2 (\alpha > 0, \beta > 0)$ the stability domain wedges into the flutter domain with the angle $\varphi_S < \pi$, Fig. 7c.

Similarly, we consider the case when at the crossing point $\alpha^2\beta^2$ at least one of the eigenvalues α or β is negative. Assuming that remaining λ are real (divergence) from (18) we find that the tangent cone K_D to the boundary of divergence domain is given by (19). This means that at this point divergence domain with an angle $\varphi_D < \pi$ wedges into flutter domain, Fig.7c.

D) Consider now a point α^3 on the curve α^2 (a cusp), see Fig. 7d. Since this point corresponds to a Jordan block of third order then according to (A34), (A35) expansions λ of due to a variation $p = p_0 + \varepsilon e$ are given by

$$\lambda = \alpha + \sqrt[3]{\langle q_1, e \rangle \varepsilon} + o\left(\varepsilon^{1/3}\right). \tag{20}$$

This formula means that for any vector e, such that $\langle q_1, e \rangle \neq 0$, a triple real α splits into one real and two complex-conjugate (flutter instability). Hence, stability domain $(\alpha > 0)$ or divergence domain $(\alpha < 0)$ at this point by a narrow tongue touches flutter domain. This picture meets the bifurcation diagram shown in Fig.6. The tangent cone to the boundary of stability or divergence domains at this point possesses zero angle, i.e. is degenerate.

Let us study what happens when a vector e_* satisfies the degeneration condition $\langle q_1, e_* \rangle = 0$. This condition defines a straight line in the plane of parameters. According to (A37), (A44) a triple real eigenvalue α splits into three simple so that two eigenvalues are

$$\lambda_1 = \alpha \pm \sqrt{\langle r_1, e_* \rangle \varepsilon} + O(\varepsilon), \tag{21}$$

while perturbation of the third eigenvalue according to (A38), (A46) is of order ε

$$\lambda_1 = \alpha + \varepsilon \frac{\langle R_1 e_*, e_* \rangle}{\langle r_1, e_* \rangle} + o(\varepsilon). \tag{22}$$

It can be seen from (22) that the third eigenvalue remains real independently of sign of e_*. But equation (22) is not enough to decide what domain does the vector e_* belong to. This can be established by analysis of expression (21). If, for example, $(r_1, e_*) > 0$, then from (21) it follows that the vector e_* lies in stability domain ($\alpha > 0$) or in divergence domain ($\alpha < 0$), while the vector -e_* lies in flutter domain because of this vector corresponds to a pair of complex-conjugate λ.

We see that there exists only one direction given by (21) in the plane of parameters so that triple real eigenvalue α splits into three simple real eigenvalues due to variation of parameters in this direction. Hence, at the point α^3 the degenerate tangent cone to the stability ($\alpha > 0$) or divergence ($\alpha < 0$) domains has the form

$$K_{S,D} = \left\{ e \in R^2 : \langle q_1, e \rangle = 0, \quad \langle r_1, e \rangle > 0 \right\}.$$

E) Now we investigate a boundary between stability and divergence domains. System (1) loses stability statically when with a change of parameters one of simple positive eigenvalues passes through zero and becomes negative. A boundary between stability and divergence domains is defined by zero eigenvalue $\lambda = 0$ which corresponds to the equation $\det A = 0$. This equation in a general case defines a smooth curve or several curves. A boundary between stability and divergence domains is a part of the set, defined by equation $\det A = 0$, since other (nonzero) eigenvalues can be arbitrary on this set.

Consider now a point on the boundary between stability and divergence domains. A simple eigenvalue $\lambda = 0$ according to (A4), (A11) changes due to a variation $p = p_0 + \varepsilon e$ as

$$\lambda = \varepsilon \langle g_1, e \rangle + o(\varepsilon) \tag{23}$$

The cone $\langle g_1, e \rangle > 0$ provides positive eigenvalues while the cone $\langle g_1, e \rangle < 0$ corresponds to negative eigenvalues. Therefore, the vector g_1 is a normal vector to the boundary between stability and divergence domains lying in the stability domain, Fig.7e. Convexity or concavity properties of this boundary can be established taking a tangent vector e_* such that $\langle g_1, e_* \rangle = 0$ and calculating the second variation λ_2 according to (A12). Then we obtain $\lambda = \varepsilon^2 \langle E_1 e_*, e_* \rangle + o(\varepsilon^2)$. Thus, when $\langle E_1 e_*, e_* \rangle > 0$ the boundary of stability domain is concave, and when $\langle E_1 e_*, e_* \rangle < 0$ it is convex. The case $\langle E_1 e_*, e_* \rangle = 0$ needs further investigation.

We remind that to calculate the vector g_1 and the matrix E_1 it is necessary to know the left and right eigenvectors corresponding to the simple eigenvalue $\lambda = 0$, and derivatives of the matrix A with respect to parameters p_1 and p_2 taken at the boundary, see (A9) - (A13).

F) A singularity of a boundary of the stability domain in the generic case also appears at the points where the curves $\alpha^2 (\alpha > 0)$ and $\beta = 0$ intersect, Fig.7f. Using (15), (23) we find the tangent cone to the stability domain at this point as

$$K_S = \left\{ e \in R^2 : \langle f_1, e \rangle > 0, \quad \langle g_1, e \rangle > 0 \right\}.$$

The angle φ_S of the tangent cone is less then π, Fig.7f.

G) Consider now a singularity $\alpha^2 (\alpha = 0)$ appearing at some points of the curve α^2. To study this singular point we have to take a variation of parameters in the form $p = p_0 + \varepsilon e + \varepsilon^2 d$ with $d \neq 0$. According to (15) we have

$$\lambda = \pm \sqrt{\langle f_1, e \rangle \varepsilon} + O(\varepsilon). \tag{24}$$

All the vectors satisfying the inequality $\langle f_1, e \rangle > 0$ lie in divergence domain, while the vectors satisfying $\langle f_1, e \rangle < 0$ belong to flutter domain.

Consider now a vector e_* tangent to the boundary, i.e. $\langle f_1, e_* \rangle = 0$. Then using (A27), (A29) we get

$$\lambda = \frac{- \langle b_1, e_* \rangle \pm \sqrt{D(e_*) + 4 \langle f_1, d \rangle}}{2} \varepsilon, \tag{25}$$

where $D = \langle b_1, e_* \rangle^2 - 4 \langle H_1 e_*, e_* \rangle$. It is easy to see from (25) that for given e_* and properly chosen vector d there exist curves $p = p_0 + \varepsilon e_* + \varepsilon^2 d$ so that double zero eigenvalue splits along these curves into two positive simple eigenvalues (stability). Different cases can occur:

1. $D(e_*) > 0, \quad \langle H_1 e_*, e_* \rangle > 0$. In this case the curves satisfying the inequalities $\langle h_1, e_* \rangle < 0$ and $-(1/4)\, D(e_*) < \langle f_1, d \rangle < \langle H_1 e_*, e_* \rangle$ lie in the stability domain. The stability domain belongs to both halfplanes with respect to the tangent vector e_*, and flutter domain is convex, see Fig.7g.

2. $D(e_*) > 0, \quad \langle H_1 e_*, e_* \rangle < 0$. In this case the curves satisfying the inequalities $\langle h_1, e_* \rangle < 0$ and $-(1/4)\, D(e_*) < \langle f_1, d \rangle < \langle H_1 e_*, e_* \rangle < 0$ and lie in the stability domain. The stability domain belongs to the halfplane $\langle f_1, d \rangle < 0$ with respect to the tangent vector e_*, and flutter domain is convex, see Fig.7g.

3. $D(e_*) < 0$. In this case the curves satisfying the inequalities $\langle h_1, e_* \rangle < 0$ and $0 < -(1/4)\, D(e_*) < \langle f_1, d \rangle < \langle H_1 e_*, e_* \rangle$ and lie in the stability domain. The stability domain belongs to the halfplane $\langle f_1, d \rangle > 0$ with respect to the tangent vector e_*, and divergence domain is convex, see Fig.7g.

Therefore, the point $\alpha^2 (\alpha = 0)$ in all three cases is common for the boundaries between three domains: divergence, flutter, and stability, the stability domain having a sharp tongue, Fig.7g. The degenerate tangent cone to the stability domain at this point takes the form

$$K_S = \left\{ e \in R^2 : \langle f_1, e \rangle = 0, \quad \langle h_1, e \rangle < 0 \right\}. \tag{26}$$

In the generic case the vectors h_1 and f_1 are linearly independent. It should be noted that to investigate the singularity in case G we have to take a variation $p = p_0 + \varepsilon e + \varepsilon^2 d$, where $d \neq 0$, since there might be no vectors $p = p_0 + \varepsilon e$ belonging to the stability domain.

Singularities, considered above in cases A-G are *generic singularities* for two-parameter circulatory systems. These singularities are inherent and don't disappear for small changes of matrix families $A(p_1, p_2)$. A hypothetical division of the parameter plane p_1, p_2, to stability, flutter, and divergence domains is presented in Fig.8.

The results obtained we formulate as a theorem

Theorem 2 *In two-parameter circulatory systems boundaries between stability, flutter, and divergence domains in the generic case are smooth curves, described in A, B, E; at some points of these curves singularities, given in C, D, F, G and shown in Fig.7, Fig.8, take place.*

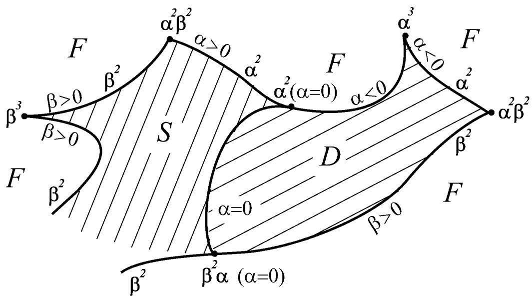

Fig. 8. Hypothetical division of the plane of parameters into stability, flutter, and divergence domains

3 Example: stability of a rigid panel subjected to an airflow

As an example, consider stability of a rigid panel of infinite span subjected to an airflow. It is assumed that the panel is maintained on two elastic supports with the stiffness coefficients c_1 and c_2 per unit span. The panel has two degrees of freedom: a vertical displacement y and a rotation φ, Fig.9.

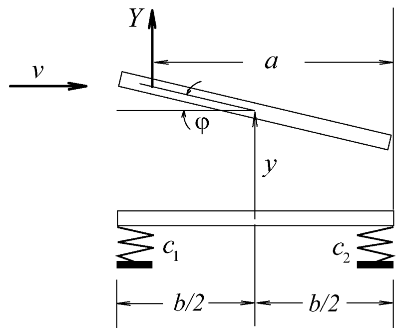

Fig. 9. A rigid panel vibrating in airflow

It is assumed that aerodynamic lift force Y acting on the panel of unit span is proportional to an angle of attack φ, to dynamic pressure of airflow, and to a width b of the panel

$$Y = c_y^\alpha \frac{\rho v^2}{2} b\varphi.$$

Here c_y^α is the partial derivative of the aerodynamic lift force coefficient with respect to the angle of attack, ρ and v are density and speed of the flow, respectively. Damping forces are not involved in the model considered. The differential equations of motion are, see [13]

$$\ddot{y} + a_{11}y + a_{12}\varphi = 0$$

$$\ddot{\varphi} + a_{21}y + a_{22}\varphi = 0 \tag{27}$$

where

$$a_{11} = \frac{c_{1+}c_2}{mb}; \qquad a_{12} = \frac{c_{1-}c_2}{2m} - c_y^\alpha \frac{\rho v^2}{2m};$$

$$a_{21} = \frac{6(c_{1-}c_2)}{mb^2}; \quad a_{22} = \frac{3(c_{1+}c_2)}{mb} + 6c_y^\alpha \frac{\rho v^2}{2}\frac{(b-2a)}{mb^2} \tag{28}$$

Here mb is mass of the panel per unit span, a is a distance between the right edge of the panel and the aerodynamic focus (the point where aerodynamic lift force is applied). Introducing dimensionless variables

$$c = \frac{c_{1-}c_2}{2(c_{1+}c_2)}, \quad \gamma = 6\left(\frac{2a}{b} - 1\right),$$

$$q = \frac{1}{2}\frac{c_y^\alpha \rho v^2}{(c_{1+}c_2)}, \quad y_* = \frac{y}{b}, \quad \tau = t\sqrt{\frac{c_{1+}c_2}{mb}},$$

and separating time with $\begin{pmatrix} y_* \\ \varphi \end{pmatrix} = ue^{\nu\tau}$, the eigenvalue problem $Au = \lambda u$ is obtained, where

$$A = \begin{bmatrix} 1 & c-q \\ 12c & 3-\gamma q \end{bmatrix}, \quad \lambda = -\nu^2. \tag{29}$$

The corresponding characteristic equation is

$$\lambda^2 + \lambda(\gamma q - 4) + 12cq - \gamma q - 12c^2 + 3 = 0. \tag{30}$$

Here q is a load parameter ($q \geq 0$), c is a stiffness parameter ($|c| \leq 0.5$), and γ characterizes the point where aerodynamic lift force is applied. For thin profile in incompressible flow $\gamma = 3$, since $a/b = 0.75$.

When parameter γ is fixed we can consider the problem as two-parametric and depending only on c and q. Taking $\gamma = 3$ characteristic equation (30) yields

$$\lambda^2 + (3q - 4)\lambda + 12cq - 3q - 12c^2 + 3 = 0, \tag{31}$$

$$\lambda_{1,2} = \frac{4 - 3q \pm \sqrt{9q^2 - 12q - 48cq + 48c^2 + 4}}{2}. \tag{32}$$

Setting $\lambda = 0$ in characteristic equation (31) we obtain a critical divergence load q_d

$$q_d(c) = \frac{1 - 4c^2}{1 - 4c}. \tag{33}$$

Since $|c| \leq 1/2, q \geq 0$, the curve $q_d(c)$ belongs to the region $c < 1/4$. Substitution of (33) into equations (31), (32) gives expressions describing how the roots λ_1,λ_2 change along this curve

$$\lambda_1(c) \equiv 0, \quad \lambda_2(c) = \frac{1 - 16c + 12c^2}{1 - 4c}. \tag{34}$$

It is easy to see that if $c < \left(4 - \sqrt{13}\right)/6$ then λ_2 is positive, and the curve $q_d(c)$ is the boundary between stability and divergence domains. In the case when inequalities $\left(4 - \sqrt{13}\right)/6 < c < 1/4$ hold the curve $q_d(c)$ belongs to divergence region since λ_2 is negative. This part of the curve (33) is shown in Fig.10 by dotted line. At the point $p_0 = \left(c = \left(4 - \sqrt{13}\right)/6, \quad q = 4/3\right)$ the root λ_2 changes the sign.

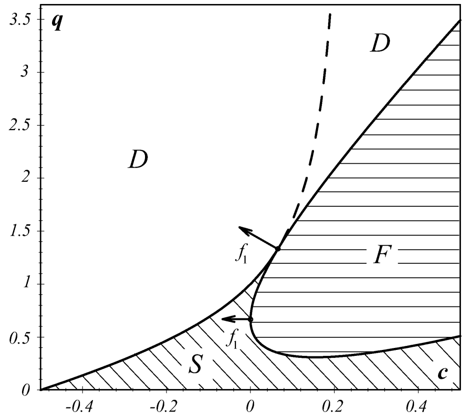

Figure 10. Stability, flutter, and divergence domains for a vibrating panel

Now we proceed to study flutter domain. It can be seen from (32) that flutter occurs when the discriminant of quadratic equation (31) becomes negative. Setting the discriminant equal to zero, a critical flutter load q_f is found as

$$q_f(c) = \frac{2}{3}\left(1 + 4c \pm 2\sqrt{c(c+2)}\right). \tag{35}$$

It follows from (35) that flutter domain belongs to the region $c \geq 0$. The curve $q_f(c)$ consists of two branches corresponding to different signs in (35). Recall that flutter boundary is characterized by double eigenvalues. From (32), (35) it can be shown that on flutter boundary $\lambda_{1,2} = (4 - 3q)/2$. Thus, if $q > 4/3$ then (35) describes the boundary between flutter and divergence domains. In the case $q < 4/3$ the curve $q_f(c)$ is the boundary between stability and flutter domains, Fig.10.

Consider now a point (c, q) at the flutter boundary. Solving the eigenvalue problem with (30) and $y = 3$ we get the vectors u_0, u_1, v_0, v_1 corresponding to the double eigenvalue $\lambda_{1,2} = (4 - 3q)/2$

$$u_0 = \begin{pmatrix} 2\dfrac{q-c}{3q-2} \\ 1 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 12c \\ -\dfrac{1}{2}(3q-2) \end{pmatrix},$$

$$u_1 = \begin{pmatrix} 0 \\ -2\dfrac{1}{3q-2} \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{36}$$

Using (36) in (A20) yields

$$f_1 = \begin{pmatrix} 12(c-q) \\ -12c + \frac{3}{2}(3q-2) \end{pmatrix}. \tag{37}$$

Consider for example the point $(c = 0,\ q = 2/3)$, where the tangent direction to the flutter boundary is vertical, Fig.10. Then according to (37) the normal vector to the boundary at this point is $f_1 = (-8;\ 0)^T$, and it lies in the stability domain, being parallel to the $c-$axis, Fig.10.

The point $p_0 = \left(c = \left(4 - \sqrt{13}\right)/6, \quad q = 4/3\right)$ corresponds to the double zero eigenvalue $\lambda_1 = \lambda_2 = 0$. At this point the flutter domain touches the divergence domain

$$q_f = q_d, \quad \frac{dq_f}{dc} = \frac{dq_d}{dc}.$$

With the use of expressions (36) we get eigenvectors and associated vectors corresponding to the double zero eigenvalue

$$u_0 = \begin{pmatrix} \frac{1}{6}(4+\sqrt{13}) \\ 1 \end{pmatrix}, \quad v_0 = \begin{pmatrix} 2(4-\sqrt{13}) \\ -1 \end{pmatrix},$$

$$u_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{38}$$

From (37) we obtain the vector $f_1 = \left(-4\sqrt{13};\ -5 + 2\sqrt{13}\right)^T$. This vector lying in the second quadrant is normal to the flutter boundary and belongs to the divergence domain, Fig.10. From the orthogonality condition $\langle f_1, e_* \rangle = 0$ we obtain the vector e_* tangent to the boundary and lying in the first quadrant

$$e_* = \begin{pmatrix} \frac{1}{52}(26 - 5\sqrt{13}) \\ 1 \end{pmatrix}. \tag{39}$$

Due to a variation of the vector of parameters $p = p_0 + \varepsilon e_*$ double zero eigenvalue splits into two simple in accordance with (A23), (A27).

Coefficients of quadratic equation (A27) with the use of (38), (39) and according to (A28), (A3) are

$$a_1 = 3, \quad a_2 = \frac{81}{52}.$$

Thus, at the singular point p_0 for the vector e_* tangent to the flutter boundary we obtain

$$a_2 > 0, \quad D = a_1^2 - 4a_2 > 0.$$

This means that the double zero eigenvalue due to the variation $p = p_0 + \varepsilon e_*$ splits in accordance with (25) into two negative simple eigenvalues λ

$$\lambda = 3\varepsilon \left(-\frac{1}{2} \pm \frac{1}{\sqrt{13}} \right) + o(\varepsilon),$$

and into two positive eigenvalues $-\lambda$ for the vector $-e_*$. Therefore, the tangent vector εe_* for rather small ε lies in divergence domain while the vector $-\varepsilon e_*$ belongs to the stability domain. The point p_0 corresponding to the double zero eigenvalue is common for three domains: divergence, flutter, and stability; flutter domain being convex. At this point the stability domain touches by a narrow tongue flutter and divergence domains, Fig.10. This result is in accordance with the general analysis, given in Section 2 of this paper.

Note that singularities α^3, $\alpha^2\beta^2$, and $\alpha^2\beta(\beta = 0)$ cannot appear in this example because of the system considered has only two degrees of freedom. Indeed, singularities α^3 and $\alpha^2\beta(\beta = 0)$ can appear in systems with three or more degrees of freedom, and $\alpha^2\beta^2$ needs at least four degrees of freedom.

4 CONCLUSION

In Sections 1, 2 stability of circulatory systems with finite degrees of freedom smoothly depending on two real parameters is studied. It is shown that in the generic case boundaries between stability, flutter and divergence domains in the plane of parameters consist of smooth curves corresponding either to double real eigenvalues with Jordan chain of second order or to simple zero eigenvalues. At some points the boundaries possess singularities, corresponding to multiple eigenvalues with Jordan chains, listed in Theorem 2 and shown in Fig.7, Fig.8. This result is based on the work of Galin (1972) where bifurcation diagrams for real matrices depending on one, two and three parameters are given. Thus, the proven Theorems supply an engineer or applied mathematician by information about nonsmoothness of stability boundaries and list all possible types of singularities and their properties in the generic (unavoidable) case.

As an example in Section 3 a problem of stability of a rigid panel subjected to airflow is considered. Stability, flutter, and divergence domains for this system are analyzed. It is shown that in this problem the singularity $\alpha^2(\alpha = 0)$ arises. Normal vectors to the stability boundary are calculated and the degenerate tangent cone at the point of singularity is found.

A method allowing to determine local geometrical properties of a boundary is developed. This method analyses bifurcations of eigenvalues near a boundary between different domains due to variation of the vector of parameters. It is based on the perturbation theory of eigenvalues of nonsymmetric matrices depending on parameters, see [19], [16]. In Appendix first- and second-order perturbation terms of simple eigenvalues are obtained. Explicit formulae describing splitting of double and triple eigenvalues with the corresponding Jordan chains of second and third order in the first approximation are given. It is shown that knowing a critical eigenvalue with the corresponding eigenvectors and associated vectors at the boundary point and using first order derivatives of the matrix of the circulatory system with respect to parameters we can construct a normal vector to the boundary at the given point. This allows us to find directions in the plane of parameters stabilizing or destabilizing the system, i.e. to construct a tangent cone to the boundary. Degenerate cases when the vector of variation of parameters is tangent to a boundary are also considered. Explicit expressions describing splitting of eigenvalues due to perturbations of parameters taken in such directions are presented. The study of degenerate cases is necessary to establish convexity properties of stability, flutter, and divergence domains at the boundary points and to find the degenerate tangent cones. Thus, using only information at a boundary point (regular or singular) we can construct approximation of the stability boundary in the vicinity of this point.

The method presented is general and can be used for investigation of local geometrical properties of the boundaries when dimension of the parameter space is arbitrary, and when singularities are not generic. The approach developed in this paper can directly lead to a computer program, hence the possible loss in stability for a high order matrices can be determined.

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Appendix: Bifurcations of eigenvalues

Let us consider an eigenvalue problem

$$Au = \lambda u, \tag{A1}$$

where A is a real nonsymmetric matrix $m \times m$ smoothly depending on a vector of real parameters $p = (p_1, p_2, ..., p_n)^T$, λ is an eigenvalue, and u is a corresponding eigenvector of dimension m.

It is assumed that λ_0 is an eigenvalue of $A(p_0)$ at fixed $p = p_0$, and a change of the eigenvalue λ is sought that depends on a change of the vector of parameters p. For this purpose, an increment $p = p_0 + \varepsilon e + \varepsilon^2 d$ is given to the vector p_0, where $\varepsilon > 0$ is a small number, e and d are real vectors of the variation of dimension n. As a result, the matrix A takes an increment

$$A(p_0 + \varepsilon e + \varepsilon^2 d) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + ... \tag{A2}$$

The matrices A_0, A_1, A_2 are described by the relations

$$A_0 = A(p_0),\ A_1 = \sum_{s=1}^{n} \frac{\partial A}{\partial p_s} e_s,\ A_2 = \sum_{i=1}^{n} \frac{\partial A}{\partial p_i} d_i + \frac{1}{2} \sum_{s,t=1}^{n} \frac{\partial^2 A}{\partial p_s \partial p_t} e_s e_t. \tag{A3}$$

Due to the variation of the vector p an eigenvalue λ and an eigenvector u take increments. According to the perturbation theory of nonselfadjoint operators, developed by Vishik and Lyusternik (1960), these increments can be expressed as series of integer or fractional powers of ε, depending on Jordan structure corresponding to the eigenvalue λ_0.

Simple eigenvalue

Assume that λ_0 is a simple eigenvalue of the matrix A_0 and u_0 is a corresponding eigenvector. In this case λ and u are smooth functions of ε and can be expressed as Taylor series ([19])

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon \lambda_1 + \varepsilon^2 \lambda_2 + ... \\ u &= u_0 + \varepsilon w_1 + \varepsilon^2 w_2 + ... \end{aligned} \tag{A4}$$

Substitution of expansions (A2), (A4) into equation (A1) yields

$$(A_0 - \lambda_0 I)u_0 = 0$$

$$(A_0 - \lambda_0 I)w_1 = \lambda_1 u_0 - A_1 u_0 \tag{A5}$$

$$(A_0 - \lambda_0 I)w_2 = \lambda_1 w_1 - A_1 w_1 + \lambda_2 u_0 - A_2 u_0.$$

Here I is the identity matrix. We introduce the inner product $(a, b) = \sum_{i=1}^{m} a_i \bar{b}_i$ for vectors $a, b \in C^m$, where the overbar denotes complex conjugate. Then the operator $L^* = A_0^T - \bar{\lambda}_0 I$ is adjoint to $L = A_0 - \lambda_0 I$ as it satisfies the relation $(Lu, v) = (u, L^* v)$ holding for arbitrary vectors u and v. For the following presentation the ad-joint eigenvalue problem is introduced

$$\left(A_0^T - \bar{\lambda}_0 I\right) v_0 = 0. \tag{A6}$$

It is assumed that the eigenvector v_0 satisfies the normality condition

$$(u_0, v_0) = 1. \tag{A7}$$

For given u_0 this equality defines v_0 uniquely. For the perturbed vector u the normality condition

$$(u, v_0) = 1 \tag{A8}$$

is used. This condition is necessary to uniquely determine all the terms λ_i and w_i.

The solvability conditions for the second and the third of equations (A5) require that the right hand sides be orthogonal to the solution of homogeneous adjoint problem (A6). Thus, equalities $(\lambda_1 u_0 - A_1 u_0, v_0) = 0$ and $(\lambda_2 u_0 - A_2 u_0 + \lambda_1 w_1 - A_1 w_1, v_0) = 0$ with the use of normality conditions (A7), (A8) give

$$\lambda_1 = (A_1 u_0, v_0), \tag{A9}$$

$$\lambda_2 = (A_2 u_0, v_0) + (A_1 w_1, v_0). \tag{A10}$$

Here $w_1 = G_0(\lambda_1 u_0 - A_1 u_0)$ is found from the second of equations (A5), and G_0 is an operator inverse to $A_0 - \lambda_0 I$, see Vishik and Lyusternik (1960).

For the sake of convenience we introduce the inner product $\langle a, b \rangle = \sum_{s=1}^{n} a_s b_s$ of vectors $a, b \in R^n$, in the parameter space. With the use of expressions (A3) for the matrices A_1 and A_2 first- and second-order eigenvalue perturbations (A9) and (A10) can be written in the form, see Seyranian (1993)

$$\lambda_1 = \langle g_1, e \rangle + i \langle g_2, e \rangle , \tag{A11}$$

$$\lambda_2 = \langle E_1 e, e \rangle + i \langle E_2 e, e \rangle + \langle g_1, d \rangle + i \langle g_2, d \rangle , \tag{A12}$$

where i is the imaginary unit. Vectors g_1 and g_2 are gradient vectors of real and imaginary parts of λ at $p = p_0$, and their components are defined by

$$g_1^s + i g_2^s = \left(\frac{\partial A}{\partial p_s} u_0, v_0 \right), \; s = 1, 2, ..., n, \tag{A13}$$

while matrices E_1 and E_2, defined by (A10), are real matrices associated with the quadratic forms on components of the vector e.

Double eigenvalue

Consider the case of a double eigenvalue λ_0 with length of Jordan chain equal to 2. This means that at $p = p_0$ there exist an eigenvector u_0 and an associated vector u_1, corresponding to λ_0 and governed by equations

$$(A_0 - \lambda_0 I) u_0 = 0$$

$$(A_0 - \lambda_0 I) u_1 = u_0. \tag{A14}$$

For the adjoint eigenvalue problem we have

$$(A_0^T - \bar{\lambda}_0 I) v_0 = 0$$

$$(A_0^T - \bar{\lambda}_0 I) v_1 = v_0. \tag{A15}$$

The vectors u_0, u_1, v_0, v_1 are related by the following conditions

$$(u_0, v_0) = 0, \quad (u_1, v_0) \equiv (u_0, v_1) \neq 0, \tag{A16}$$

that can be proved by means of equations (A14) and (A15), see [19].

Taking a variation of the vector of parameters $p = p_0 + \varepsilon e + \varepsilon^2 d$, the variation of the matrix A in the form (A2) results. In the case of a Jordan block expansions for eigenvalues and eigenvectors contain terms with fractional powers $\varepsilon^{j/l}$, $j = 0, 1, 2, ...$, where l is length of Jordan chain ([19]). The case of a double eigenvalue λ_0 with $l = 2$ yields

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + ... \\ u &= u_0 + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \varepsilon^{3/2}w_3 + ... \end{aligned} \tag{A17}$$

Substituting (A17) and (A2) into (A1) , we get equations for determining coefficients of expansions for the eigenvalue and the eigenvector

$$(A_0 - \lambda_0 I)w_1 = \lambda_1 u_0$$

$$(A_0 - \lambda_0 I)w_2 = -A_1 u_0 + \lambda_2 u_0 + \lambda_1 w_1. \tag{A18}$$

Assuming that the vector u_0 is fixed, the normality conditions for v_0,v_1 and u are used in the form

$$(u_0, v_1) = 1, \quad (u, v_1) = 1. \tag{A19}$$

From the first of equations (A18) using (A14) it is seen that $w_1 = G_0(\lambda_1 u_0) = \lambda_1 G_0 u_0 = \lambda_1 u_1 + \gamma u_0$. The second of normality conditions (A19) yields for unknown constant $\gamma = -\lambda_1(u_1, v_1)$. Then, the solvability condition for the second of equations (A18) with (A16), (A19) gives

$$\lambda_1^2 = (A_1 u_0, v_0). \tag{A20}$$

If the right hand side of (A20) is not zero, it yields two different nonzero roots $\lambda_1 = \pm\sqrt{(A_1 u_0, v_0)}$ with the corresponding vectors $w_1 = \lambda_1(u_1 - (u_1, v_1)u_0)$. Using expression (A3) for the matrix A_1, equation (A20) can be re-written in the form, see [16]

$$\lambda_1^2 = \langle f_1, e\rangle + i \langle f_2, e\rangle . \tag{A21}$$

Components of the vectors f_1 and f_2 are defined by

$$f_1^s + i f_2^s = \left(\frac{\partial A}{\partial p_s} u_0, v_0 \right), \ s = 1, 2, ..., n. \tag{A22}$$

Equation (A21) describes splitting of a double eigenvalue with Jordan chain of second order in the general case $(A_1 u_0, v_0) \neq 0$.

Consider now a *degenerate* case when the right hand side of (A20) is zero. In the sequel we assume that the vector of variation e_* belongs to the hyperplane defined by the degeneration condition $(A_1 u_0, v_0) = \langle f_1, e_* \rangle + i \langle f_2, e_* \rangle = 0$. Then it is easy to verify that the terms with fractional powers in a general case disappear in (A17), and we have

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon \lambda_2 + \varepsilon^2 \lambda_4 + ... \\ u &= u_0 + \varepsilon w_2 + \varepsilon^2 w_4 + ... \end{aligned} \tag{A23}$$

Substituting (A23), (A2) into (A1) , we obtain equations for determining coefficients of expansions

$$(A_0 - \lambda_0 I) w_2 = \lambda_2 u_0 - A_1 u_0$$

$$(A_0 - \lambda_0 I) w_4 = \lambda_2 w_2 - A_1 w_2 + \lambda_4 u_0 - A_2 u_0. \tag{A24}$$

The solvability condition for the second of equations (A24) with (A16) yields

$$\lambda_2 (w_2, v_0) - (A_1 w_2, v_0) - (A_2 u_0, v_0) = 0. \tag{A25}$$

The term (w_2, v_0) can be found from the first of equations (A24) with the use of (A15) and normality condition (A19)

$$(w_2, v_0) = \lambda_2 - (A_1 u_0, v_1). \tag{A26}$$

Substituting (A26) and $w_2 = G_0(\lambda_2 u_0 - A_1 u_0)$ into (A25) leads to the quadratic equation on λ_2

$$\lambda_2^2 + \lambda_2 a_1 + a_2 = 0 \tag{A27}$$

with the coefficients

$$a_1 = -(A_1 u_0, v_1) - (A_1 u_1, v_0),$$

$$a_2 = -(A_2 u_0, v_0) + \left(G_0(A_1 u_0), A_1^T v_0\right). \tag{A28}$$

Equation (A27) defines in the first approximation splitting of a double eigenvalue λ_0 in the degenerate case. With the use of expressions (A3) for the matrices A_1,A_2 the coefficients a_1 and a_2 can be given in the form

$$a_1 = \langle b_1, e_* \rangle + i \langle b_2, e_* \rangle,$$

$$a_2 = \langle H_1 e_*, e_* \rangle + i \langle H_2 e_*, e_* \rangle - \langle f_1, d \rangle - i \langle f_2, d \rangle. \tag{A29}$$

Here the real vectors b_1,b_2 and the real matrices H_1,H_2 are defined by expressions (A28).

Triple eigenvalue

Consider the case of a triple eigenvalue that is characterized by Jordan chain of length 3. This means that there are eigenvectors u_0, v_0 and associated vectors u_1, u_2, v_1, v_2 satisfying the equations for main and adjoint eigenvalue problems

$$(A_0 - \lambda_0 I)u_0 = 0, \quad (A_0^T - \bar{\lambda}_0 I)v_0 = 0,$$

$$(A_0 - \lambda_0 I)u_1 = u_0, \quad (A_0^T - \bar{\lambda}_0 I)v_1 = v_0, \tag{A30}$$

$$(A_0 - \lambda_0 I)u_2 = u_1, \quad (A_0^T - \bar{\lambda}_0 I)v_2 = v_1.$$

The vectors u_i, v_i are related by the following equalities

$$(u_0, v_0) = (u_1, v_0) = (u_0, v_1) = 0, \tag{A31}$$

$$(u_2, v_1) = (u_1, v_2), \quad (u_2, v_0) = (u_1, v_1) = (u_0, v_2). \tag{A32}$$

These conditions can be proved by means of equations (A30). Assuming that the vectors u_0, u_1, u_2 are fixed, the normality conditions for v_0, v_1,v_2 and u are used

$$(u_0, v_2) = 1, \quad (u, v_2) = 1. \tag{A33}$$

Due to perturbation of parameters $p = p_0 + \varepsilon e + \varepsilon^2 d$ the triple eigenvalue λ_0 generally splits into three simple eigenvalues λ. The eigenvalues and the eigenvectors are given in the form ([19])

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/3}\lambda_1 + \varepsilon^{2/3}\lambda_2 + \varepsilon\lambda_3 + ... \\ u &= u_0 + \varepsilon^{1/3}w_1 + \varepsilon^{2/3}w_2 + \varepsilon w_3 + ... \end{aligned} \tag{A34}$$

In the same way as for a double eigenvalue the expression for λ_1 can be found as

$$\lambda_1^3 = (A_1 u_0, v_0) \equiv \langle q_1, e\rangle + i\,\langle q_2, e\rangle\,, \tag{A35}$$

where the real vectors q_1 and q_2 are defined by the relation

$$q_1^s + iq_2^s = \left(\frac{\partial A}{\partial p_s} u_0, v_0\right), \; s = 1, 2, ..., n. \tag{A36}$$

If $(A_1 u_0, v_0) \neq 0$, then expression (A35) yields three different complex roots $\lambda_1 = \sqrt[3]{(A_1 u_0, v_0)}$, describing splitting of the triple eigenvalue with Jordan chain of third order.

In the *degenerate* case when the right hand side of (A35) is zero the triple eigenvalue splits into three simple by another way. Expansions for two of the eigenvalues and for the corresponding eigenvectors contain terms with fractional powers $\varepsilon^{j/2}$, $j = 1, 2, ...$

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon^{1/2}\lambda_1 + \varepsilon\lambda_2 + \varepsilon^{3/2}\lambda_3 + ... \\ u &= u_0 + \varepsilon^{1/2}w_1 + \varepsilon w_2 + \varepsilon^{3/2}w_3 + ..., \end{aligned} \tag{A37}$$

while expansions for the third of eigenvalues and for the corresponding eigenvector contain only integer powers of ε, see [19]:

$$\begin{aligned} \lambda &= \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + ... \\ u &= u_0 + \varepsilon w_1 + \varepsilon^2 w_2 + ... \end{aligned} \tag{A38}$$

Substituting (A37), (A2) into (A1), we obtain

$$(A_0 - \lambda_0 I)w_1 = \lambda_1 u_0,$$

$$(A_0 - \lambda_0 I)w_2 = \lambda_1 w_1 - A_1 u_0 + \lambda_2 u_0, \tag{A39}$$

$$(A_0 - \lambda_0 I)w_3 = \lambda_1 w_2 - A_1 w_1 + \lambda_2 w_1 + \lambda_3 u_0.$$

The first of equations (A39) gives $w_1 = G_0(\lambda_1 u_0) = \lambda_1 u_1 + \gamma u_0$, where $\gamma = -\lambda_1(u_1, v_2)$ is found from (A33). Then solvability condition for the third of equations (A39) with (A31) yields

$$(w_2, v_0) = (A_1 u_1, v_0). \tag{A40}$$

Taking the inner product of the second of equations (A39) with v_1 and using (A31), (A32), (A33), and $w_1 = \lambda_1 u_1 + \gamma u_0$, gives

$$\lambda_1^2 = (w_2, v_0) + (A_1 u_0, v_1). \tag{A41}$$

After substitution of (A40) into (A41) we obtain

$$\lambda_1^2 = (A_1 u_1, v_0) + (A_1 u_0, v_1). \tag{A42}$$

Introducing real vectors r_1 and r_2, whose components are defined by

$$r_1^s + i r_2^s = \left(\frac{\partial A}{\partial p_s} u_1, v_0\right) + \left(\frac{\partial A}{\partial p_s} u_0, v_1\right), \; s = 1, 2, ..., n, \tag{A43}$$

we have from (A42)

$$\lambda_1 = \pm\sqrt{\langle r_1, e_* \rangle + i \langle r_2, e_* \rangle}. \tag{A44}$$

In the same way, substituting (A38), (A2) into (A1), for the third eigenvalue we obtain the expression for determining λ_1 in expansions (A38) in the form

$$\lambda_1 = \frac{\left(G_0(A_1 u_0), A_1^T v_0\right) - (A_2 u_0, v_0)}{(A_1 u_1, v_0) + (A_1 u_0, v_1)}. \tag{A45}$$

This formula can be written as

$$\lambda_1 = \frac{\langle R_1 e_*, e_*\rangle + i\,\langle R_2 e_*, e_*\rangle - \langle q_1, d\rangle - i\,\langle q_2, d\rangle}{\langle r_1, e_*\rangle + i\,\langle r_2, e_*\rangle}, \tag{A46}$$

where R_1 and R_2 are real matrices defined by the equality

$$\langle R_1 e_*, e_*\rangle + i\,\langle R_2 e_*, e_*\rangle = \big(G_0(A_1 u_0), A_1^T v_0\big) -$$

$$- \left(\left(\frac{1}{2} \sum_{s,t=1}^{n} \frac{\partial^2 A}{\partial p_s \partial p_t} e_*^s e_*^t \right) u_0, v_0 \right).$$

Equations (A44), (A46) describe splitting of the triple eigenvalue λ_0 in the degenerate case $(A_1 u_0, v_0) \equiv \langle q_1, e_*\rangle + i\,\langle q_2, e_*\rangle = 0$ under condition $\langle r_1, e_*\rangle + i\,\langle r_2, e_*\rangle \neq 0$.

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Bifurkacioni dijagrami i granice stabilnosti cirkulatornih sistema

UDK 531.36

Razmatraju se problemi stabilnosti linearnih cirkulatornih sistema opšteg tipa sa konačnim brojem stepeni slobode koji zavise od dva parametra. Pokazano je da su ovi sistemi u izvornom slučaju izloženi flaterskim i divergentnim nestabilnostima. Bifurkacije sopstvenih vrednosti koje opisuju mehanizam statičkih i dinamičkih gubitaka stabilnosti se studiraju i daje se geometrijska interpretacija ovih katastrofa. Za dvodimenzioni slučaj granice izmedju domena stabilnosti, flatera i divergencije se analiziraju. Tangentni konusi i vektori normalni na granice se izračunavaju pomocu levih i desnih sopstvenih i pridruženih vektora. Kao primer se razmatra stabilnost krutog panela koji osciluje u struji vazduha.