# EFFECT OF SMALL INTERNAL AND EXTERNAL DAMPING ON THE STABILITY OF CONTINUOUS NON-CONSERVATIVE SYSTEMS

**Oleg N. Kirillov**
Institute of Mechanics
Moscow State Lomonosov University, Moscow
Russia
kirillov@imec.msu.ru

**Alexander P. Seyranian**
Institute of Mechanics
Moscow State Lomonosov University, Moscow
Russia
seyran@imec.msu.ru

**Abstract**

An effect of small internal and external damping on the stability of continuous non-conservative systems is investigated. A theory is developed, qualitatively and quantitatively describing the destabilization paradox in non-conservative systems, i.e. the jump in the critical load and frequency of the system when small dissipative forces are taken into account. The theory is based on the bifurcation analysis of multiple eigenvalues of non-self-adjoint boundary eigenvalue problems depending on parameters. It is shown that the destabilization paradox is related to the perturbation of the double eigenvalue of a circulatory system by small damping. The formulae are derived, which describe the behavior of eigenvalues of a non-conservative system due to change of the load and damping parameters. Explicit expressions for the jumps in the critical load and frequency are found. Stabilization conditions for small damping are established. As a mechanical example the stability of a viscoelastic rod with small internal and external damping, loaded by tangential follower force, is studied in detail.

**Key words**

Nonconservative system, non-self-adjoint boundary eigenvalue problem, destabilization paradox, damping, multiple eigenvalue

## 1 Introduction

Studying the stability of a two-link pendulum loaded by a follower force [Ziegler, 1952] came to an unexpected conclusion that the critical load of the non-conservative system with vanishingly small damping is considerably lower than that of the system without dissipation. This effect known as *the destabilization paradox* was detected in many other non-conservative mechanical systems, both discrete and continuous, see e.g. [Bolotin, 1963], [Bolotin and Zhinzher, 1969], [Bolotin et al., 2002], [Andreichikov and Yudovich, 1974], [Panovko and Sorokin, 1987], [Seyranian, 1990], [Seyranian, 1996], [Seyranian and Pedersen, 1993], [Zhinzher, 1994], [Langthjem and Sugiyama, 2000]. In recent papers by [Seyranian and Kirillov, 2003] and [Kirillov, 2004] the effect of small velocity-dependent forces on the stability of finite-dimensional non-conservative systems was studied in general formulation.

In this contribution we analyze the stabilizing and destabilizing effect of small damping for rather general class of continuous non-conservative systems, described by the non-self-adjoint boundary eigenvalue problems. Explicit asymptotic expressions obtained for the stability domain allow us to predict when a given combination of the damping parameters leads to increase or to decrease in the critical non-conservative load. The results obtained explain why different types of internal and external damping so surprisingly influence on the stability of non-conservative systems.

## 2 Bifurcation of multiple eigenvalues

To start our analysis we consider generalized boundary eigenvalue problem for the non-self-adjoint differential operator smoothly dependent on the complex spectral parameter and vector of real parameters. The study of bifurcations of multiple eigenvalues of the operator due to change of the parameters provides us by necessary information on the system stability.

### 2.1 Basic relations

Following [Mennicken and Möller, 2003] we denote by $L$ a linear differential operator of order $m$ with respect to the variable $x$. The action of the operator on a smooth function $u(x)$ is defined by the expression

$$Lu = \sum_{j=0}^{m} l_j \frac{d^{m-j}u}{dx^{m-j}} \tag{1}$$

The coefficients $l_j(x, \lambda, \mathbf{p})$ of the operator $L$ smoothly depend on $x$. The function $l_0(x)$ defined on the interval $x \in [0, 1]$ is bounded from the bottom by a positive constant. It is assumed that the coefficients $l_j(x, \lambda, \mathbf{p})$ depend analytically on a complex spectral parameter $\lambda$ and are smooth functions of the vector of real parameters $\mathbf{p} \in \mathbb{R}^n$.

We define the matrix of boundary conditions as the block matrix $\mathbf{U} = [\mathbf{A}\ \mathbf{B}]$ of dimension $m \times 2m$ and rank $m$, consisting of the $m \times m$ blocks $\mathbf{A}$ and $\mathbf{B}$. It is convenient to define the vector $\mathbf{u} = (\mathbf{u}(0), \mathbf{u}(1))$ of dimension $2m$, where the vectors $\mathbf{u}(0) = (u(0), u'_x(0), \ldots, u_x^{(m-1)}(0))$ and $\mathbf{u}(1) = (u(1), u'_x(1), \ldots, u_x^{(m-1)}(1))$ consist of the values of the function $u(x)$ and its derivatives evaluated at the boundary points $x = 0$ and $x = 1$. Then,

$$\mathbf{U}\mathbf{u} = \mathbf{A}\mathbf{u}(0) + \mathbf{B}\mathbf{u}(1) \tag{2}$$

It is assumed that the entries of the matrices $\mathbf{A}(\lambda, \mathbf{p})$ and $\mathbf{B}(\lambda, \mathbf{p})$ are analytical functions of the complex spectral parameter $\lambda$ and smoothly depend on the vector of real parameters $\mathbf{p} \in \mathbb{R}^n$.

On the interval $x \in [0, 1]$ we consider the eigenvalue problem for the differential operator $L$ with the boundary conditions defined by the matrix $\mathbf{U}$

$$L(x, \lambda, \mathbf{p})u = 0, \ \ \mathbf{U}(\lambda, \mathbf{p})\mathbf{u} = 0 \tag{3}$$

If the functions $y_1(x)$, $y_2(x)$, $\ldots$, $y_m(x)$ form a fundamental system of solutions of the differential equation (3), then its general solution has the form $u(x) = \sum_{j=1}^{m} c_j y_j(x)$. A nontrivial solution of the boundary problem (3) exists if and only if the characteristic determinant is zero

$$\det(\mathbf{A}\mathbf{Y}(0){+}\mathbf{B}\mathbf{Y}(1)){=}0 \tag{4}$$

The entries of the matrix $\mathbf{Y}(x)$ are defined by the expressions $Y_{ij}(x){=}{y_j}_x^{(i-1)}(x)$, $i, j = 1, 2, \ldots, m$.

For a fixed vector $\mathbf{p}{=}\mathbf{p}_0$ a value $\lambda_0$ of the spectral parameter, for which a nontrivial solution $u_0$ of the problem (3) exists, is an eigenvalue, and a function $u_0$ is an eigenfunction, corresponding to $\lambda_0$. Eigenvalues of the problem (3) are the roots of the equation (4).

Let us introduce a block matrix $\widetilde{\mathbf{U}}{=}[\widetilde{\mathbf{A}}\ \widetilde{\mathbf{B}}]$ of dimension $m{\times}2m$, where the matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ of dimension $m{\times}m$ in the general case depend on the spectral parameter $\lambda$ and vector of real parameters $\mathbf{p}$. We choose the matrices $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ so that the block matrix of dimension $2m \times 2m$, composed of the matrices $\mathbf{U}$, $\widetilde{\mathbf{U}}$, is nonsingular in the vicinity of the point $\mathbf{p} = \mathbf{p}_0$ and the eigenvalue $\lambda = \lambda_0$. Then, we define the matrices $\mathbf{V}$ and $\widetilde{\mathbf{V}}$ of dimension $m \times 2m$ by the expressions

$$\begin{bmatrix} -\widetilde{\mathbf{V}} \\ \mathbf{V} \end{bmatrix}^* = \begin{bmatrix} -\mathbf{L}(0) & \mathbf{O} \\ \mathbf{O} & \mathbf{L}(1) \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \end{bmatrix}^{-1} \tag{5}$$

where $\mathbf{O}$ is the matrix of dimension $m \times m$ with zero entries, and the asterisk indicates Hermitian conjugation.

The matrices $\mathbf{L}(0)$ and $\mathbf{L}(1)$ in (5) are the values of the $m \times m$ matrix $\mathbf{L}(x)$ at the points $x = 0$ and $x = 1$. The entries $L_{ij}(x)$ of the matrix are expressed by means of the coefficients of the differential operator $L$ and their derivatives with respect to $x$

$$L_{ij}(x) = \sum_{k=i-1}^{m-j} (-1)^k C_k^{i-1} \frac{d^{k-i+1}}{dx^{k-i+1}} l_{m-j-k}, \tag{6}$$

$$C_k^{i-1} = \begin{cases} \frac{k!}{(i-1)!(k-i+1)!}, & k \geq i{-}1 \geq 0, \\ 0, & k < i{-}1. \end{cases} \tag{7}$$

The operator $L^*$, which is adjoint to $L$, is defined by the expression [Mennicken and Möller, 2003]

$$L^* v = \sum_{j=0}^{m} (-1)^{m-j} \frac{d^{m-j}}{dx^{m-j}} (\overline{l_j(x)} v) \tag{8}$$

The eigenvalue problem adjoint to (3) is

$$L^*(\bar{\lambda}, \mathbf{p})v = 0, \ \ \mathbf{V}(\bar{\lambda}, \mathbf{p})\mathbf{v} = 0 \tag{9}$$

where the vector $\mathbf{v} = (\mathbf{v}(0), \mathbf{v}(1))$ and the vectors $\mathbf{v}(0) = (v(0), v'_x(0), \ldots, v_x^{(m-1)}(0))$ and $\mathbf{v}(1) = (v(1), v'_x(1), \ldots, v_x^{(m-1)}(1))$ are composed of the values of the function $v(x)$ and its derivatives taken at the points $x{=}0$ and $x{=}1$.

### 2.2 Perturbation of eigenvalues

Now we assume that in the vicinity of the point $\mathbf{p}_0$ the spectrum of the problem (3) is discrete and contains a $\mu$-fold eigenvalue $\lambda_0$ with the Keldysh chain of length $\mu$, consisting of the eigenfunction $u_0$ and associated functions $u_1, \ldots, u_{\mu-1}$ [Keldysh, 1971], [Gohberg et al., 1982]. We denote $L_0{=}L(\lambda_0, \mathbf{p}_0)$ and $\mathbf{U}_0{=}\mathbf{U}(\lambda_0, \mathbf{p}_0)$. The functions of the Keldysh chain satisfy the boundary value problems [Mennicken and Möller, 2003]

$$L_0 u_0 = 0, \ \ \mathbf{U}_0 \mathbf{u}_0 = 0 \tag{10}$$

$$L_0 u_j {=} -\sum_{r=1}^{j} \frac{1}{r!} L_\lambda^{(r)} u_{j-r}, \ \mathbf{U}_0 \mathbf{u}_j {=} -\sum_{r=1}^{j} \frac{1}{r!} \mathbf{U}_\lambda^{(r)} \mathbf{u}_{j-r} \tag{11}$$

where the partial derivatives are evaluated at $\lambda{=}\lambda_0$ and $\mathbf{p}{=}\mathbf{p}_0$.

The Keldysh chain of the complex-conjugate eigenvalue $\overline{\lambda}_0$ of the adjoint operator $L_0^*$, is defined by the expressions

$$L_0^* v_0 = 0, \;\; \mathbf{V}_0 \mathbf{v}_0 = 0 \tag{12}$$

$$L_0^* v_j = -\sum_{r=1}^{j} \frac{1}{r!} L^{*(r)}_{\bar{\lambda}} v_{j-r}, \;\; \mathbf{V}_0 \mathbf{v}_j = -\sum_{r=1}^{j} \frac{1}{r!} \mathbf{V}^{(r)}_{\bar{\lambda}} \mathbf{v}_{j-r} \tag{13}$$

The functions of the Keldysh chain are related by the orthogonality conditions

$$\sum_{r=1}^{j} \frac{1}{r!} \left( (L_\lambda^{(r)} u_{j-r}, v_0) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \mathbf{U}_\lambda^{(r)} \mathbf{u}_{j-r} \right) = 0 \tag{14}$$

where $(u, v) = \int_0^1 u(x) \overline{v}(x) dx$ is Hermitian inner product of functions $u$ and $v$. We note that in the formulae (11), (13), and (14) the index $j = 1, 2, \ldots, \mu - 1$.

In the $n$-dimensional parameter space we consider a smooth curve, dependent on a real parameter $\epsilon \geq 0$

$$\mathbf{p}(\epsilon) = \mathbf{p}_0 + \epsilon \dot{\mathbf{p}} + \frac{\epsilon^2}{2} \ddot{\mathbf{p}} + o(\epsilon^2) \tag{15}$$

where dot denotes differentiation with respect to $\epsilon$, and the derivatives are evaluated at $\epsilon = 0$. Due to the perturbation of parameters (15), the eigenvalue $\lambda_0$ with the Keldysh chain of length $\mu$ takes an increment represented by the Newton-Puiseux series [Vishik and Lyusternik, 1960]

$$\lambda = \lambda_0 + \lambda_1 \epsilon^{1/\mu} + \ldots + \lambda_{\mu-1} \epsilon^{(\mu-1)/\mu} + \lambda_\mu \epsilon + \ldots \tag{16}$$

The coefficient $\lambda_1$ is found by the method of small perturbations as

$$\lambda_1^\mu = -\frac{(L_1 u_0, v_0) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0}{\sum_{r=1}^{\mu} \frac{1}{r!} \left( \left( \frac{\partial^r L}{\partial \lambda^r} u_{\mu-r}, v_0 \right) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \frac{\partial^r \mathbf{U}}{\partial \lambda^r} \mathbf{u}_{\mu-r} \right)} \tag{17}$$

where

$$L_1 = \sum_{j=1}^{n} \frac{\partial L}{\partial p_j} \dot{p}_j, \;\; \mathbf{U}_1 = \sum_{j=1}^{n} \frac{\partial \mathbf{U}}{\partial p_j} \dot{p}_j \tag{18}$$

and the derivatives are evaluated at $\lambda = \lambda_0$, $\mathbf{p} = \mathbf{p}_0$, and $\epsilon = 0$. In accordance with the formulae (16) and (17) the eigenvalue $\lambda_0$ splits due to perturbation (15) to $\mu$ simple eigenvalues, if $\lambda_1 \neq 0$.

Under the constraint $\lambda_1 = 0$ the double eigenvalue ($\mu = 2$) splits according to the formula $\lambda = \lambda_0 + \epsilon \lambda_2 + o(\epsilon)$, where the coefficient $\lambda_2$ is found from the quadratic equation

$$\lambda_2^2 \sigma_2 + \lambda_2 \left( (L_1 u_0, v_1) + (L_1 u_1, v_0) + \left( \frac{\partial L_1}{\partial \lambda} u_0, v_0 \right) \right)$$

$$+ \lambda_2 \left( \mathbf{v}_1^* \widetilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_0 + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \mathbf{U}_1 \mathbf{u}_1 + \mathbf{v}_0^* \frac{\partial (\widetilde{\mathbf{V}}^* \mathbf{U}_1)}{\partial \lambda} \mathbf{u}_0 \right)$$

$$+ (L_2 u_0, v_0) + (L_1 \hat{w}_2, v_0) + (\widetilde{\mathbf{V}}_0 \mathbf{v}_0)^* (\mathbf{U}_2 \mathbf{u}_0 + \mathbf{U}_1 \hat{\mathbf{w}}_2) = 0 \tag{19}$$

where

$$\sigma_2 = \sum_{r=1}^{2} \frac{1}{r!} \left( \left( \frac{\partial^r L}{\partial \lambda^r} u_{2-r}, v_0 \right) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \frac{\partial^r \mathbf{U}}{\partial \lambda^r} \mathbf{u}_{2-r} \right) \tag{20}$$

The operator $L_2$ and the matrix $\mathbf{U}_2$ in equation (19) have the form

$$L_2 = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial L}{\partial p_j} \ddot{p}_j + \frac{1}{2} \sum_{j,t=1}^{n} \frac{\partial^2 L}{\partial p_j \partial p_t} \dot{p}_j \dot{p}_t \tag{21}$$

$$\mathbf{U}_2 = \frac{1}{2} \sum_{j=1}^{n} \frac{\partial \mathbf{U}}{\partial p_j} \ddot{p}_j + \frac{1}{2} \sum_{j,t=1}^{n} \frac{\partial^2 \mathbf{U}}{\partial p_j \partial p_t} \dot{p}_j \dot{p}_t \tag{22}$$

The function $\hat{w}_2$ is a solution to the boundary value problem

$$L_0 \hat{w}_2 = -L_1 u_0, \;\; \mathbf{U}_0 \hat{\mathbf{w}}_2 = -\mathbf{U}_1 \mathbf{u}_0 \tag{23}$$

where the vector $\hat{\mathbf{w}}_2 = (\hat{\mathbf{w}}_2(0), \hat{\mathbf{w}}_2(1))$ and

$$\hat{\mathbf{w}}_2(\xi) = (\hat{w}_2(\xi), \hat{w}_{2x}'(\xi), \ldots, \hat{w}_{2x}^{(m-1)}(\xi)), \;\; \xi = 0, 1 \tag{24}$$

Hence, we obtained explicit formulae describing splitting of multiple eigenvalues of the non-self-adjoint boundary eigenvalue problem (3) with a change of parameters in the general and degenerate cases.

## 3 Non-conservative systems with small damping

Let us formulate the boundary eigenvalue problem arising in stability problems for viscoelastic systems

$$L(\lambda, q, \mathbf{k}) u \equiv N(q) u + \lambda D(\mathbf{k}) u + \lambda^2 M u = 0 \tag{25}$$

$$\mathbf{U}(q, \mathbf{k}, \lambda) \mathbf{u} \equiv \mathbf{U}_N(q) \mathbf{u} + \lambda \mathbf{U}_D(\mathbf{k}) \mathbf{u} + \lambda^2 \mathbf{U}_M \mathbf{u} = 0 \tag{26}$$

The coefficients of the differential operators $N$, $D$, and $M$ of order $m$, and of the matrices $\mathbf{U}_N$, $\mathbf{U}_D$, and

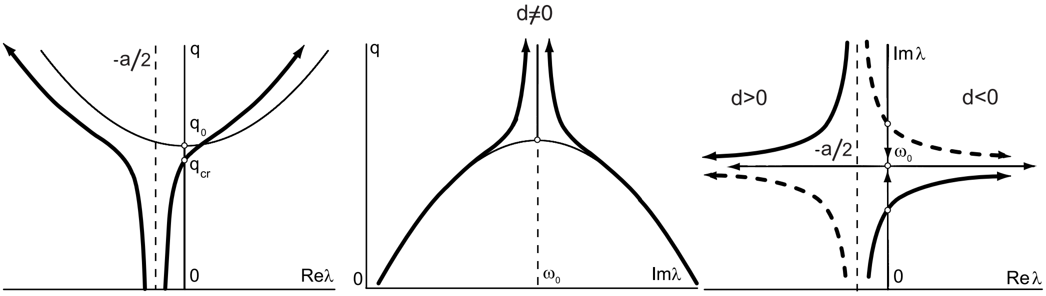


Figure 1. Trajectories of eigenvalues for the undamped (fine lines) and damped (bold lines) systems

$\mathbf{U}_M$ of dimension $m \times 2m$ are assumed to be real. The operator $N$ and the matrix $\mathbf{U}_N$ smoothly depend on the real "load" parameter $q \geq 0$. The coefficients of the differential operator $D$ and the matrix $\mathbf{U}_D$ are smooth functions of the vector of real "damping" parameters $\mathbf{k} = (k_1, \ldots, k_{n-1})$, and $\mathbf{k} = 0$ yields $D(0) = 0$, $\mathbf{U}_D(0) = 0$. The operator $M$ and the matrix $\mathbf{U}_M$ are parametrically independent. We assume that the perturbation of the non-conservative system by small dissipative forces ($\|\mathbf{k}\| \ll 1$) is regular [Vishik and Lyusternik, 1960] and does not increase the order of operator.

Let the unperturbed circulatory system

$$N(q)u + \lambda^2 Mu = 0, \;\; \mathbf{U}_N(q)\mathbf{u} + \lambda^2 \mathbf{U}_M \mathbf{u} = 0 \quad (27)$$

have discrete spectrum for $0 \leq q < q_0$, consisting of simple purely imaginary eigenvalues $\lambda$ (stability). At $q = q_0$ there exists a pair of double eigenvalues $\pm i\omega_0$, $\omega_0>0$ with the Keldysh chain of length 2, which yields flutter instability [Bolotin, 1963], [Kirillov and Seyranian, 2004]. All remaining eigenvalues $\pm i\omega_{0,j}$, $\omega_{0,j} > 0$ of the unperturbed system at $q = q_0$ are assumed to be simple and purely imaginary. Therefore, in the absence of dissipative forces ($\mathbf{k} = 0$) the value $q = q_0$ is the boundary between the stability and flutter domains.

The eigenfunction $u_0$ and associated function $u_1$ of the eigenvalue $i\omega_0$ satisfy the equations (10) and (11). The functions $v_0$ and $v_1$ of the adjoint Keldysh chain are solutions of equations (12) and (13) with $\mu$=2. Note that the functions $u_0$ and $v_0$ are defined up to arbitrary multipliers and the functions $u_1$ and $v_1$ are defined up to the terms $\gamma_1 u_0$, $\gamma_2 v_0$, respectively, where $\gamma_1$ and $\gamma_2$ are arbitrary coefficients. We choose the real functions $u_0$ and $v_0$, and the imaginary associated functions $u_1$ and $v_1$, which satisfy the orthonormality conditions

$$2i\omega_0(Mu_1, v_1) + (Mu_0, v_1) + (Mu_1, v_0) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \mathbf{U}_M \mathbf{u}_1$$

$$+(\widetilde{\mathbf{V}}_0 \mathbf{v}_1 + \widetilde{\mathbf{V}}'_\lambda \mathbf{v}_0)^* (2i\omega_0 \mathbf{U}_M \mathbf{u}_1 + \mathbf{U}_M \mathbf{u}_0) = 0 \quad (28)$$

$$2i\omega_0((Mu_1, v_0) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \mathbf{U}_M \mathbf{u}_1) = 1 \quad (29)$$

This choice is possible, because the matrices $\mathbf{V}_0$ and $\widetilde{\mathbf{V}}_0$, defined by the expression (5) for $\lambda = i\omega_0$, $\mathbf{k}$=0 and $q$=$q_0$, are real, and the matrices $\frac{\partial \mathbf{V}}{\partial \lambda}(\bar{\lambda}_0, \mathbf{p}_0)$ and $\frac{\partial \widetilde{\mathbf{V}}}{\partial \overline{\lambda}}(\bar{\lambda}_0, \mathbf{p}_0)$ have imaginary entries. This can be verified by expressing the inverse of the matrix polynomial in formula (5) according to the extended Leverrier algorithm [Barnett, 1989], [Wang and Lin, 1993]

Studying the splitting of the double eigenvalue $i\omega_0$ of the problem (25), (26) due to small variation of the parameters $\mathbf{k}$ and $q$ according to the formulae (16)–(24) we find equations describing the movement of the eigenvalues on the complex plane

$$(\mathrm{Im}\lambda - \omega_0 + \mathrm{Re}\lambda + a/2)^2$$

$$-(\mathrm{Im}\lambda - \omega_0 - \mathrm{Re}\lambda - a/2)^2 = -2d \quad (30)$$

$$\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^4 + \left(c - \frac{a^2}{4}\right)\left(\mathrm{Re}\lambda + \frac{a}{2}\right)^2 = \frac{d^2}{4} \quad (31)$$

$$(\mathrm{Im}\lambda - \omega_0)^4 - \left(c - \frac{a^2}{4}\right)(\mathrm{Im}\lambda - \omega_0)^2 = \frac{d^2}{4} \quad (32)$$

The quantities $a$, $c$, and $d$ are determined by the expressions

$$a = -\,\omega_0 \langle \mathbf{h}, \mathbf{k} \rangle, \;\; c = \widetilde{f}(q - q_0) + \omega_0^2 \langle \mathbf{G}\mathbf{k}, \mathbf{k} \rangle$$

$$d = \omega_0 \left( \langle \mathbf{f}, \mathbf{k} \rangle + \langle \mathbf{H}\mathbf{k}, \mathbf{k} \rangle \right) \quad (33)$$

where the angular brackets denote the inner product of vectors in $\mathbb{R}^{n-1}$. The components of the real vector $\mathbf{f}$ and the real quantity $\widetilde{f}$ are

$$\widetilde{f} = \left( \frac{\partial N}{\partial q} u_0, v_0 \right) + \mathbf{v}_0^* \widetilde{\mathbf{V}}_0^* \frac{\partial \mathbf{U}_N}{\partial q} \mathbf{u}_0 \quad (34)$$

$$f_r=\left(\frac{\partial D}{\partial k_r}u_0, v_0\right)+\mathbf{v}_0^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0 \qquad (35)$$

The components of the real vector $\mathbf{h}$ are defined by the expressions

$$ih_r=\left(\frac{\partial D}{\partial k_r}u_1, v_0\right)+\left(\frac{\partial D}{\partial k_r}u_0, v_1\right)$$

$$+\mathbf{v}_1^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0+\mathbf{v}_0^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_1+\mathbf{v}_0^*\left(\frac{\partial \widetilde{\mathbf{V}}}{\partial \bar{\lambda}}\right)^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0 \qquad (36)$$

The entries of the real matrix $\mathbf{H}$ are

$$H_{r\sigma}=\frac{1}{2}\left(\frac{\partial^2 D}{\partial k_r \partial k_\sigma}u_0, v_0\right)+\frac{1}{2}\mathbf{v}_0^*\widetilde{\mathbf{V}}_0^*\frac{\partial^2 \mathbf{U}_D}{\partial k_r \partial k_\sigma}\mathbf{u}_0 \qquad (37)$$

and the real matrix $\mathbf{G}$ is determined by the expression

$$\langle \mathbf{G}\dot{\mathbf{k}}, \dot{\mathbf{k}}\rangle=\sum_{r=1}^{n-1}\dot{k}_r\left(\left(\frac{\partial D}{\partial k_r}\hat{w}_2, v_0\right)+\mathbf{v}_0^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\hat{\mathbf{w}}_2\right) \qquad (38)$$

The function $\hat{w}_2$ is a solution to the boundary value problem

$$N(q_0)\hat{w}_2-\omega_0^2 M\hat{w}_2=\sum_{r=1}^{n-1}\dot{k}_r\frac{\partial D}{\partial k_r}u_0 \qquad (39)$$

$$\mathbf{U}_N(q_0)\hat{\mathbf{w}}_2-\omega_0^2\mathbf{U}_M\hat{\mathbf{w}}_2=\sum_{r=1}^{n-1}\dot{k}_r\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_0 \qquad (40)$$

For the circulatory system we have $\mathbf{k}=0$ and in accordance with expression (33) the quantities $a=0$, $c=\widetilde{f}(q-q_0)$, and $d=0$. Then, equations (31), (32) yield

$$q\le q_0: \ \ \mathrm{Re}\lambda=0, \ \ \mathrm{Im}\lambda=\omega_0\pm\sqrt{\widetilde{f}(q-q_0)} \qquad (41)$$

$$q\ge q_0: \ \ \mathrm{Re}\lambda=\pm\sqrt{-\widetilde{f}(q-q_0)}, \ \ \mathrm{Im}\lambda=\omega_0 \qquad (42)$$

As it follows from equations (41) and (42) for $\widetilde{f}<0$ and increasing load parameter $q$ two simple purely imaginary eigenvalues move along the imaginary axis until they collide at $q=q_0$. After the collision the eigenvalues diverge in the directions perpendicular to the imaginary axis of the complex plane, forming a complex-conjugate pair (flutter) as shown in Figure 1. Such a scenario is known as the strong interaction of eigenvalues and is a typical mechanism of the loss of stability for circulatory systems [Seyranian and Mailybaev, 2003], [Kirillov and Seyranian, 2004].

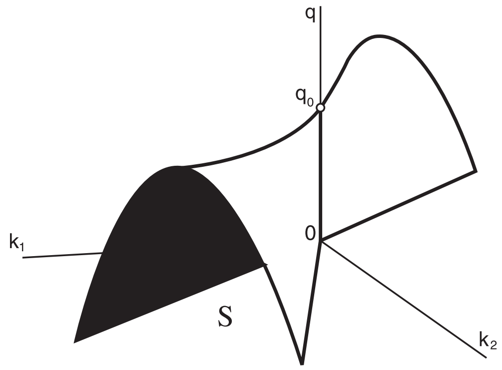


Figure 2. Whitney umbrella singularity of the critical load surface

Introduction of damping ($\mathbf{k}\neq 0$) changes the instability mechanism. With the variation of the parameter $q$ and under the condition $d\neq 0$ the eigenvalues move separately along the branches of hyperbola (30) on the complex plane Figure 1. The hyperbola has two asymptotes $\mathrm{Re}\lambda=-a/2$ and $\mathrm{Im}\lambda=\omega_0$, where the quantity $a$ is given by the first of equations (33). When $a>0$, one of the two eigenvalues is in the left side of the complex plane, while another one crosses the imaginary axis and goes to the right side at $q=q_{cr}(\mathbf{k})$. Thus, the condition $a>0$ or, equivalently, $\langle \mathbf{h}, \mathbf{k}\rangle<0$ is necessary for asymptotic stability. The critical value $q_{cr}$ of the load parameter follows from equation (31) under the assumption $\mathrm{Re}\lambda=0$. This yields the relation $ca^2=d^2$, which with the use of the explicit expressions (33) for $a$, $c$, and $d$ takes the form

$$q_{cr}(\mathbf{k})=q_0+\frac{(\langle \mathbf{f}, \mathbf{k}\rangle+\langle \mathbf{Hk}, \mathbf{k}\rangle)^2}{\widetilde{f}\langle \mathbf{h}, \mathbf{k}\rangle^2}-\frac{\omega_0^2}{\widetilde{f}}\langle \mathbf{Gk}, \mathbf{k}\rangle \qquad (43)$$

Therefore, the two eigenvalues are in the left side of the complex plane, if $q<q_{cr}(\mathbf{k})$ and $\langle \mathbf{h}, \mathbf{k}\rangle<0$.

Asymptotic stability of the system (25), (26) also depends on the behavior of the simple eigenvalues $\pm i\omega_{0,s}$, $\omega_{0,s}>0$ with a change of parameters. These eigenvalues move to the left side of the complex plane under the conditions

$$\langle \mathbf{g}_s, \mathbf{k}\rangle>0, \ s=1,2,\ldots \qquad (44)$$

where the components of the real vector $\mathbf{g}_s$ are

$$g_{s,r}=\left(\frac{\partial D}{\partial k_r}u_{0,s}, v_{0,s}\right)+\mathbf{v}_{0,s}^*\widetilde{\mathbf{V}}_0^*\frac{\partial \mathbf{U}_D}{\partial k_r}\mathbf{u}_{0,s} \qquad (45)$$

It is assumed that the eigenfunctions of the simple eigenvalues satisfy the normalization conditions

$$2\omega_{0,s}(Mu_{0,s}, v_{0,s})=1, \ \ s=1,2,\ldots \qquad (46)$$

In the case when $\{\mathbf{k} : \langle \mathbf{f}, \mathbf{k}\rangle = 0, \langle \mathbf{h}, \mathbf{k}\rangle < 0\} \subset \{\mathbf{k} : \langle \mathbf{g}_s, \mathbf{k}\rangle > 0, s = 1, 2, \ldots\}$ small perturbation of parameters $q$ and $\mathbf{k}$ shifts all simple eigenvalues $\pm i\omega_{0,s}$ to the left from the imaginary axis. Hence, the stability of the system (25), (26) is determined only by the splitting of the double eigenvalues $\pm i\omega_0$. In this case the surface $q_{cr}(k_1, \ldots, k_{n-1})$ approximated by equation (43) is the boundary of the asymptotic stability domain in the vicinity of the point $\mathbf{p}_0 = (0, \ldots, 0, q_0)$.

If the vector of damping parameters $\mathbf{k}$ consists of only two components $k_1$ and $k_2$, the surface $q_{cr}(k_1, k_2)$ given by the expression (43) has a Whitney umbrella singularity at the point $(0, 0, q_0)$ of the space of parameters $k_1, k_2, q$, see Figure 2.

The level sets of the function (43) are the boundaries of the stability domain in the space of the damping parameters $\mathbf{k}=(k_1, \ldots, k_{n-1})$. In particular, the level set $q_{cr}=q_0$, where $q_0$ is the critical load of the circulatory system is given by the expression

$$\langle \mathbf{f}, \mathbf{k}\rangle = \pm\omega_0 \langle \mathbf{h}, \mathbf{k}\rangle \sqrt{\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle} - \langle \mathbf{H}\mathbf{k}, \mathbf{k}\rangle \tag{47}$$

Equation (47) has real solutions, if $\langle \mathbf{G}\mathbf{k}, \mathbf{k}\rangle \geq 0$. Then, the set (47) bounds the domain where the variation of the vector of the damping parameters yields $q_{cr}(\mathbf{k}) > q_0$. This means stabilization of the non-conservative system by small damping forces.

As it is clear from equation (43), the function $q_{cr}(\mathbf{k})$ is singular at the point $\mathbf{k} = 0$, and the critical load as a function of $n - 1$ variable has no limit as $\mathbf{k} = (k_1, \ldots, k_{n-1})$ tends to zero. However, homogeneity of the numerator and denominator of the rational part of $q_{cr}(\mathbf{k})$ guarantees the existence of the limit $\lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}})$ for any direction $\widetilde{\mathbf{k}}$ such that $\langle \mathbf{h}, \widetilde{\mathbf{k}}\rangle \neq 0$. Substituting $\mathbf{k} = \epsilon\widetilde{\mathbf{k}}$ into equation (43) and taking the limit we find an explicit expression approximating *the jump in the critical load* due to small damping

$$\Delta q \equiv q_0 - \lim_{\epsilon\to 0} q_{cr}(\epsilon\widetilde{\mathbf{k}}) = -\frac{1}{\widetilde{f}} \frac{\langle \mathbf{f}, \widetilde{\mathbf{k}}\rangle^2}{\langle \mathbf{h}, \widetilde{\mathbf{k}}\rangle^2} \tag{48}$$

Substitution of the expression $\mathrm{Re}\lambda=0$ into equation (30) yields approximation of *the jump in the critical frequency* caused by small damping $\mathbf{k}=\epsilon\widetilde{\mathbf{k}}$

$$\Delta\omega \equiv \omega_0 - \lim_{\epsilon\to 0} \omega_{cr}(\epsilon\widetilde{\mathbf{k}}) = -\frac{\langle \mathbf{f}, \widetilde{\mathbf{k}}\rangle}{\langle \mathbf{h}, \widetilde{\mathbf{k}}\rangle} \tag{49}$$

Thus, for $\langle \mathbf{f}, \widetilde{\mathbf{k}}\rangle=0$ the jumps in the critical load and frequency do not happen ($\Delta q=0, \Delta\omega=0$). According to equation (43) the critical load in this case tends to $q_0$ as $\epsilon \to 0$. However, the function $q_{cr}(\epsilon\widetilde{\mathbf{k}})$ can decrease for $\langle \mathbf{f}, \widetilde{\mathbf{k}}\rangle=0$, if $\langle \mathbf{G}\widetilde{\mathbf{k}}, \widetilde{\mathbf{k}}\rangle < 0$. For $\langle \mathbf{G}\widetilde{\mathbf{k}}, \widetilde{\mathbf{k}}\rangle > 0$ the critical load is increasing $q_{cr}(\epsilon\widetilde{\mathbf{k}}) \geq q_0$ (stabilization).

## 4 Stability of a viscoelastic rod with different types of external damping

As an example we consider transverse vibrations of a cantilevered rod about vertical equilibrium position. The rod is made of the viscoelastic Kelvin-Voight material with the damping coefficient $\eta \geq 0$. It is assumed that the rod is loaded by the tangential follower force $q$ at its free end, as shown in Figure 4. In the following, we will treat two cases, which differ only by the type of the external damping force applied to the rod.

### 4.1 External damping due to resistance of a viscous medium

Let us additionally assume that the rod is vibrating in a viscous medium with the damping coefficient $\mu \geq 0$. Investigation of stability of this system is reduced to the study of the boundary eigenvalue problem written in non-dimensional variables [Andreichikov and Yudovich, 1974]

$$(1 + \eta\lambda)u''''_{xxxx} + qu''_{xx} + (\lambda^2 + \mu\lambda)u = 0 \tag{50}$$

$$u(0)=0, \;\; u'_x(0)=0, \;\; u''_{xx}(1)=0, \;\; u'''_{xxx}(1)=0 \tag{51}$$

When the damping is absent ($\eta = \mu = 0$), the elastic rod is stable for the follower loads in the interval $0 \leq q < q_0$, where $q_0$=20.05 [Beck, 1952]. At $q = q_0$ the spectrum of the problem (50), (51) is discrete and consists of the pair of the double eigenvalues $\pm i\omega_0$ ($\omega_0$=11.02), other eigenvalues $\pm i\omega_{0,s}$, $s = 1, 2, \ldots$ being simple and purely imaginary. The first members of the sequence of simple eigenfrequencies are

$$\omega_{0,1} = 53.71, \;\; \omega_{0,2} = 112.4, \;\; \omega_{0,3} = 191.1 \ldots \tag{52}$$

Higher frequencies demonstrate the asymptotic behavior [Andreichikov and Yudovich, 1974]

$$\omega_{0,s} = \pi^2 s^2 + O(s), \;\; s \to \infty \tag{53}$$

The eigenfunctions of the problem (50), (51) in the undamped case are found in [Bolotin and Zhinzher, 1969]

$$u_{0,s}(x)= \cosh(ax)-\cos(bx)+F(a \sin(bx)-b \sinh(ax)) \tag{54}$$

$$v_{0,s}(x)= \cosh(ax)-\cos(bx)+G(a \sin(bx)-b \sinh(ax)) \tag{55}$$

where

$$F=\frac{a^2 \cosh(a) + b^2 \cos(b)}{ab(a \sinh(a) + b \sin(b))} \tag{56}$$

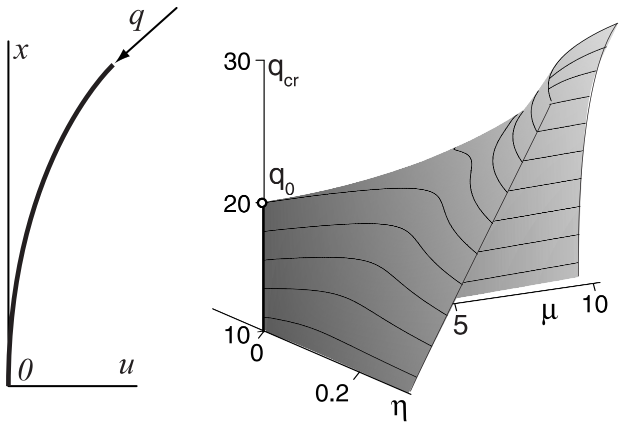


Figure 3. Approximation (62), (63) of the asymptotic stability domain of a viscoelastic rod in a viscous medium

$$G=\frac{b^2\cosh(a)+a^2\cos(b)}{b^3\sinh(a)+a^3\sin(b)} \tag{57}$$

and the coefficients $a$ and $b$ are

$$a=\sqrt{-\frac{q}{2}+\sqrt{\frac{q^2}{4}+\omega_{0,s}^2}},\ \ b=\sqrt{\frac{q}{2}+\sqrt{\frac{q^2}{4}+\omega_{0,s}^2}} \tag{58}$$

For high eigenfrequencies the eigenfunctions are represented by the asymptotic expressions [Andreichikov and Yudovich, 1974]

$$u_{0,s}=\sin(s\pi x)+O(s^{-1})$$

$$v_{0,s}=\sin(s\pi x)+O(s^{-1}),\ \ s\to\infty \tag{59}$$

Substitution of the eigenvalues (52), (53) and the corresponding eigenfunctions (54), (55) into equations (45) yields the vectors $\mathbf{g}_s$

$$\mathbf{g}_1=(35.44,0.009),\ \ \mathbf{g}_2=(65.03,0.004),\ \ \ldots \tag{60}$$

$$\mathbf{g}_s=\frac{1}{2}\left(s^2\pi^2+o(s^2),s^{-2}\pi^{-2}+o(s^{-2})\right),\ \ s\to\infty \tag{61}$$

Inequalities (44) with the vector $\mathbf{k}=(\eta,\mu)$ and the vectors $\mathbf{g}_s$ determined by (60) and (61) give the conditions for the simple eigenvalues to be in the left side of the complex plane.

Substitution of the quantities (34)–(38) computed with the use of the eigenfunctions and associated functions of the double eigenvalue of the problem (50), (51) without damping found in [Kirillov and Seyranian, 2004] into the formula (43) yields approximation of the critical load as a function of damping parameters

$$q_{cr}(\eta,\mu)=q_0-\frac{1902\eta^2}{(14.34\eta+0.091\mu)^2}+12.68\eta\mu+0.053\mu^2 \tag{62}$$

Combining the stability conditions given by simple and double eigenvalues we find that the viscoelastic rod in viscous medium is asymptotically stable in the vicinity of the point $\eta=0$, $\mu=0$, $q=q_0$, if the following inequalities are satisfied

$$\eta>0,\ \ \mu>-157.9\eta,\ \ q<q_{cr}(\eta,\mu) \tag{63}$$

Note that for positive damping parameters the first two conditions (63) are satisfied. Approximation of the asymptotic stability domain (63) in the space of the parameters $\eta$, $\mu$, $q$ is shown in Figure 3. One can see the Whitney umbrella singularity at the point $\eta=0$, $\mu=0$, $q=q_0$. Figure 3 clearly shows the existence of the domain where $q_{cr}(\eta,\mu)>q_0$. Indeed, for $\eta=0$ from equation (62) we have $q_{cr}=q_0+0.053\mu^2\geq q_0$. We can conclude that small internal ($\eta$) and external ($\mu$) damping can stabilize the viscoelastic rod loaded by the follower force.

For $\mu=0$ and $\eta\to 0$ expression (62) gives the critical load $q_{cr}=10.8<q_0=20.05$, which is in a good agreement with the value $q_{cr}=10.94$ numerically found in [Bolotin and Zhinzher, 1969], [Andreichikov and Yudovich, 1974]. Thus, formula (62) gives a good approximation to the jump in the critical load due to small damping.

### 4.2 External damping due to a dash-pot

In this case we neglect the influence of the damping due to resistance of the medium. Instead, we assume that a dash-pot with the damping coefficient $\delta$ is attached to the free end of the rod, Figure 4. This system is governed by the boundary eigenvalue problem [Panovko and Sorokin, 1987], [Zhinzher, 1994]

$$(1+\eta\lambda)u''''_{xxxx}+qu''_{xx}+\lambda^2u=0 \tag{64}$$

$$u(0)=u'_x(0)=0,\ \ u''_{xx}(1)=(1+\eta\lambda)u'''_{xxx}(1)-\delta\lambda u(1)=0 \tag{65}$$

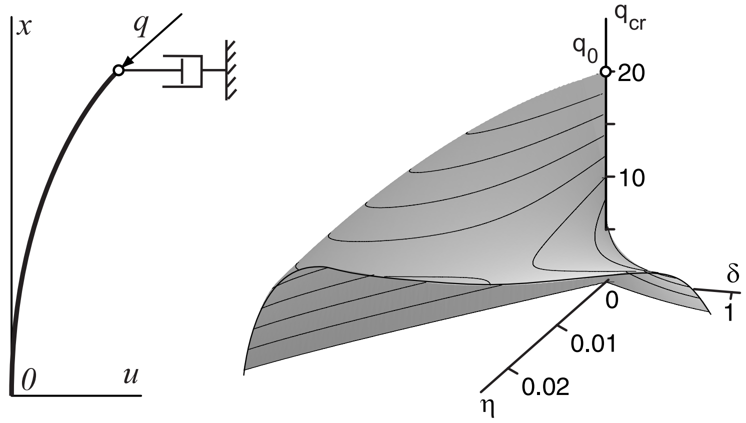


Figure 4.  Approximation (69), (70) of the asymptotic stability domain of a viscoelastic rod with a dash-pot at the free end

Substituting the eigenvalues (52), (53) and the corresponding eigenfunctions into equations (45) we get the vectors $\mathbf{g}_s$

$$\mathbf{g}_1 = (35.44, 0.043), \ \ \mathbf{g}_2 = (65.03, 0.020), \ldots \quad (66)$$

$$\mathbf{g}_s = \frac{1}{2}\left(s^2\pi^2 + o(s^2), o(s^{-2})\right), \ \ s \to \infty \quad (67)$$

Inequalities (44) with the vector $\mathbf{k} = (\eta, \delta)$ and the vectors $\mathbf{g}_s$ determined by (66) and (67) give the conditions for the simple eigenvalues to be in the left side of the complex plane

$$\eta > 0, \ \ \delta > -820.5\eta \quad (68)$$

Using in equations (34)–(38) the eigenfunctions and associated functions for the double eigenvalue $\omega_0 = 11.02$ of the undamped rod at $q = 20.05$ found in [Kirillov and Seyranian, 2004] we find the coefficients of the formula (43). Then, the critical load as a function of internal ($\eta$) and external ($\delta$) damping is

$$q_{cr}(\eta, \delta) = q_0 - \frac{(43.61\eta + 0.719\delta)^2}{(14.34\eta + 0.134\delta)^2} - 1368\eta^2 + 248.8\delta\eta \quad (69)$$

Combining the stability conditions following from the study of behavior of the simple and double eigenvalues we get the approximation of the asymptotic stability domain

$$\eta > 0, \ \ \delta > -107.0\eta, \ \ q < q_{cr}(\eta, \delta) \quad (70)$$

The stability domain described by conditions (70) is shown in Figure 4. One can see that the critical load as a function of damping parameters has a singularity at the origin. According to (69) the critical load decreases in a discontinuous manner for any combination of positive damping parameters $\eta$ and $\delta$ in the vicinity of the origin. Thus, small external damping caused by a dash-pot destabilizes the rod, contrary to the resistance of a medium, which has a stabilizing effect.

## 5 Conclusion

We have studied continuous non-conservative mechanical systems with small internal and external damping. Destabilizing and stabilizing effects of small damping are analytically described with the use of the bifurcation theory of multiple eigenvalues developed for non-self-adjoint boundary eigenvalue problems. Two mechanical examples illustrate the effectiveness of the developed approach. It is shown that small external damping caused by a dash-pot destabilizes a viscoelastic rod loaded by the follower force, while the resistance of a medium has a stabilizing effect.

## 6 Acknowledgements

The work is supported by the Alexander von Humboldt Fellowship and by the research grants RFBR 03-01-00161 and CRDF-BRHE Y1-MP-06-19.

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