Subcritical flutter in the acoustics of friction of the spinning rotationally symmetric elastic continua

O.N. Kirillov
T.U. Darmstadt, Department of Mechanical Engineering,
Hochschulstr. 1, D-64289, Darmstadt, Germany
e-mail: **kirillov@dyn.tu-darmstadt.de**

Abstract

Linearized models of elastic bodies of revolution, spinning about their symmetrical axes, possess the eigenfrequency plots with respect to the rotational speed, which form a mesh with double semi-simple eigenfrequencies at the nodes. At contact with friction pads, the rotating continua, such as the singing wine glass or the squealing disc/drum brake, start to vibrate because of the subcritical flutter instability. In the present paper a sensitivity analysis of the spectral mesh is developed for the explicit predicting the onset of instability. The determining role of the Krein signature of the eigenvalues involved in the crossings as well as the key role of the indefinite damping and non-conservative positional forces is clarified in the development and localization of the subcritical flutter. It is established that even when the rotational symmetry is broken by the variation of the structure of the stiffness matrix and therefore the eigenvalues of the undamped gyroscopic system avoid crossings, its perturbation by the dissipative forces with the indefinite matrix can cause flutter instability in the subcritical region.

1 Introduction

Consider an autonomous linear gyroscopic system describing small oscillations in the discretized models of rotating elastic bodies of revolution considered in a stationary frame [1, 2]

$$\ddot{\mathbf{x}} + 2\Omega\mathbf{G}\dot{\mathbf{x}} + (\mathbf{P} + \Omega^2\mathbf{G}^2)\mathbf{x} = 0, \quad \mathbf{x} = \mathbb{R}^{2n}, \tag{1}$$

where Ω is the speed of rotation, $\mathbf{P}$=$\mathrm{diag}(\omega_1^2, \omega_1^2, \omega_2^2, \omega_2^2, \dots, \omega_n^2, \omega_n^2) = \mathbf{P}^T$ is the matrix of potential forces, and $\mathbf{G}$=$\mathrm{diag}(\mathbf{J}, 2\mathbf{J}, \dots, n\mathbf{J}) = -\mathbf{G}^T$ is the matrix of gyroscopic forces with

$$\mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \tag{2}$$

Due to rotational symmetry of the rotor and periodic boundary conditions the eigenvalues $\omega_1^2 < \omega_2^2 < \cdots < \omega_{n-1}^2 < \omega_n^2$ of the matrix $\mathbf{P}$ are double semi-simple, that is each eigenvalue ω_s^2 has two linearly independent eigenvectors [3, 4]. The distribution of the doublets ω_s^2 as a function of s is usually different for various bodies of revolution. For example, $\omega_s = s$ corresponds to the spectrum of a circular string [5]. Nevertheless, there exist isospectral bodies, because "one cannot hear the shape of a drum" [6].

Separating time by the substitution $\mathbf{x} = \mathbf{u}\exp(\lambda t)$, we arrive at the eigenvalue problem for the operator $\mathbf{L}_0$

$$\mathbf{L}_0(\Omega)\mathbf{u} := (\mathbf{I}\lambda^2 + 2\Omega\mathbf{G}\lambda + \mathbf{P} + \Omega^2\mathbf{G}^2)\mathbf{u} = 0. \tag{3}$$

As a consequence of the block-diagonal structure of the matrices $\mathbf{G}$ and $\mathbf{P}$ the eigenvalues of $\mathbf{L}_0$ are

$$\lambda_s^{\pm} = i\omega_s \pm is\Omega, \quad \overline{\lambda_s^{\pm}} = -i\omega_s \mp is\Omega, \tag{4}$$

where bar over a symbol denotes complex conjugate. Rotation causes the doublet modes $\pm i\omega_s$ to split. The newborn pair of simple eigenvalues $\lambda_s^{\pm}$ corresponds to the forward and backward traveling waves, which propagate along the circumferential direction [7]. Viewed from the stationary frame, the frequency of the forward traveling wave appears to increase and that of the backward traveling wave appears to decrease, as the spin increases. Double eigenvalues thus originate again at non-zero angular velocities, forming the nodes of the *spectral mesh* [8, 9] of the crossed frequency curves in the plane 'frequency' versus 'angular velocity'. The spectral meshes are characteristic for such rotating symmetric continua as circular strings [5], discs [10, 11, 12], rings and cylindrical and hemispherical shells [13], vortex rings [14], and a spherically symmetric α^2-dynamo of magnetohydrodynamics [8].

At the angular velocity $\Omega_s^{cr} = \omega_s/s$ the frequency of the sth backward traveling wave vanishes to zero ($\lambda_s^{\pm} = \overline{\lambda_s^{\pm}} = 0$), so that the wave remains stationary in the non-rotating frame. The lowest one of such velocities, Ω_{cr}, is called *critical* [12]. When the speed of rotation exceeds the critical speed, the backward wave travels slower than the disc rotation speed and appears to be traveling forward (reflected wave), corresponding to the eigenvalues $\overline{\lambda_s^{\pm}}$. The effective energy of the reflected wave is negative and that of the forward and backward traveling waves is positive [15]. Therefore, in the *subcritical* speed region $|\Omega| < \Omega_{cr}$ all the crossings of the frequency curves correspond to the forward and backward modes of the same signature, while in the *supercritical* speed region $|\Omega| > \Omega_{cr}$ there exist crossings that are formed by the reflected and forward/backward modes of opposite signature. According to Krein's theory [15], under Hamiltonian perturbations like the mass and stiffness constraints [10], the crossings in the subcritical region veer away into *avoided crossings* (stability), while in the supercritical region the crossings of the modes of opposite signature turn into the rings of complex eigenvalues—*bubbles of instability* [15]—leading to flutter known also as the 'mass and stiffness instabilities' [10].

A *supercritical flutter* is important for the high speed applications like circular saws and computer storage devices, while in the acoustics of friction of rotating elastic bodies of revolution a *subcritical flutter* is either desirable as a source of instability at low speeds as in the case of musical instruments like the singing wine glass and a glass harmonica [16, 17] or undesirable as in the case of the squelaing disc- and drum brakes [12, 18, 19, 20, 21]. Being prohibited by Krein's theory for the Hamiltonian systems, subcritical flutter can occur, however, due to non-Hamiltonian, i.e. dissipative and non-conservative, perturbations [22].

The author of one of the first theories of squeal [23], Spurr experimentally observed that a rotating wine glass sang when the dynamic friction coefficient was a decreasing function of the velocity [16]. Linearizing the system with the negative friction-velocity gradient produces an eigenvalue problem with an *indefinite* matrix of damping forces. Effectively negatively damped vibration modes may lead to complex eigenvalues with positive real parts and cause flutter instability [24, 25, 26, 27]. The growth in amplitude will be limited in practice by some non-linearity. Since the engineering design is often more concerned with if a brake may squeal and less with how loud the brake may squeal, a complex eigenvalue analysis offers for it a pragmatic approach used currently by most of production work in industry [20].

The fall in the dynamic friction coefficient with increasing velocity is among the main empirical reasons for disc brake squeal, categorized in [19]. One more is *non-conservative* positional forces, which first appeared in the linear models by North [28]. The binary flutter in such models happens through the coalescence of two modes according to the reversible Hopf bifurcation scenario [27, 29, 30]. Inclusion of damping leads to the imperfect merging of modes [31] and to the flutter through the dissipative Hopf bifurcation, which is connected to the reversible one by means of the Whitney umbrella singularity [29, 32, 33]. The destabilizing role of non-potential positional forces in dynamical systems, including the tippe top inversion and the rising egg phenomena of rotordynamics, was emphasized recently in [34], see also [26, 35, 36].

Historically, in the study of brake squeal, the symmetry of the disc as well as the effects of its rotation were frequently ignored. The latter in the assumption that the low rotor speed range which squeal tends to occur does not warrant this complication [20]. However, as in the case of a singing wine glass, experiments revealed the proximity of the squealing frequency and mode shape of brake's rotor for low rotational speeds to a natural frequency and corresponding mode shape of a stationary rotor [12, 19, 20, 37, 38]. Since an axially symmetric rotor possesses pairs of identical frequencies, Chan et al. [39] proposed another mechanism of squeal in the classification of Kinkaid et al. [19] based on the *splitting of the frequency of the doublet modes* in the symmetric disc when a friction force was applied. The splitting could lead to flutter equated to brake squeal, which in general is a sound with one dominant frequency [12, 18, 19, 20, 21].

In 1990s subcritical flutter was detected by numerical approaches in the new models of disc brakes that incorporated gyroscopic and centripetal effects and accommodated more than one squeal mechanism through the splitting the doublet modes of a disc by dissipative and non-conservative perturbations coming from the negative friction-velocity gradient and frictional follower load. The models include both the case when the point-wise or distributed friction pads are rotated around a stationary disc, affecting a point or a sector of it, and when the disc rotates past the stationary friction pads, see [19, 20, 21, 39, 40], and references therein. Linearization and discretization of the latter class of the models frequently results in the equation (1) perturbed by the matrices $\mathbf{D} = \mathbf{D}^T$, $\mathbf{K} = \mathbf{K}^T$ and $\mathbf{N} = -\mathbf{N}^T$ corresponding to dissipative, potential and non-conservative positional forces

$$\ddot{\mathbf{x}} + (2\Omega\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + (\mathbf{P} + \Omega^2\mathbf{G}^2 + \kappa\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0, \tag{5}$$

where the parameters δ, κ, and ν control the magnitudes of the perturbations. The matrices $\mathbf{D}$, $\mathbf{K}$, and $\mathbf{N}$ can be assumed to be functions of Ω. The transformation $\mathbf{x} = \mathbf{A}\mathbf{z} := \exp(-\Omega\mathbf{G}t)\mathbf{z}$ yields an equivalent to (5) potential system with the periodic perturbation, see [41, 42]

$$\ddot{\mathbf{z}} + \delta\widetilde{\mathbf{D}}(t)\dot{\mathbf{z}} + (\mathbf{P} - \delta\Omega\widetilde{\mathbf{D}}(t)\widetilde{\mathbf{G}}(t) + \kappa\widetilde{\mathbf{K}}(t) + \nu\widetilde{\mathbf{N}}(t))\mathbf{z} = 0, \tag{6}$$

with $\widetilde{\mathbf{D}}(t) = \mathbf{A}^{-1}\mathbf{D}\mathbf{A}$, $\widetilde{\mathbf{G}}(t) = \mathbf{A}^{-1}\mathbf{G}\mathbf{A} = \mathbf{G}$, $\widetilde{\mathbf{K}}(t) = \mathbf{A}^{-1}\mathbf{K}\mathbf{A}$, $\widetilde{\mathbf{N}}(t) = \mathbf{A}^{-1}\mathbf{N}\mathbf{A}$, because in the rotating frame the load appears to be moving periodically in the circumferencial direction. For $2n = 2$ degrees of freedom the matrix $\widetilde{\mathbf{N}}(t) = \mathbf{N}$ and the periodic stiffness and damping matrices are

$$\begin{aligned} 2\widetilde{\mathbf{K}}(t) &= \operatorname{diag}(\mathrm{tr}\mathbf{K}, \mathrm{tr}\mathbf{K}) + (\mathbf{K} + \mathbf{J}\mathbf{K}\mathbf{J})\cos(2\Omega t) + (\mathbf{J}\mathbf{K} - \mathbf{K}\mathbf{J})\sin(2\Omega t), \\ 2\widetilde{\mathbf{D}}(t) &= \operatorname{diag}(\mathrm{tr}\mathbf{D}, \mathrm{tr}\mathbf{D}) + (\mathbf{D} + \mathbf{J}\mathbf{D}\mathbf{J})\cos(2\Omega t) + (\mathbf{J}\mathbf{D} - \mathbf{D}\mathbf{J})\sin(2\Omega t). \end{aligned} \tag{7}$$

In the absence of dissipative ($\delta = 0$) and non-conservative positional ($\nu = 0$) terms, (6) is a Mathieu-like equation with the periodic in time potential possessing parametric resonances in the supercritcal range $|\Omega| > \Omega_{cr}$. Inclusion of parametrically excited damping and non-conservative terms makes the equation (6) a less traditional parametric resonance problem due to the possibility of instability in both the subcritical and supercritical regions [21, 39]. Nevertheless, the equivalence of the two dual descriptions enables us to reduce the investigation of the parametric resonance in the non-autonomous system (6) to a considerably simpler study of the stability of the autonomous system (5), cf. [42]. Subcritical parametric resonance domains of equation (6) correspond to the regions of subcritical flutter of the system (5).

In the present paper we propose a sensitivity analysis of the system (5) based on the perturbation theory of multiple eigenvalues of non-self-adjoint operators [3, 4, 8, 43], which is an efficient tool for investigation of the subcritical flutter both in the finite-dimensional and distributed models. Instead of deriving the particular operators of dissipative and circulatory forces by accurate modeling of the frictional contact and then studying their effect on the spectrum and stability, we solve an inverse problem. Assuming *a priori* only the existence of distinct squeal frequencies close to the double eigenfrequencies of the uloaded body we find the structure of the dissipative and non-conservative operators whose action causes flutter in the subcritical region near the nodes of the spectral mesh. We describe analytically the movement of eigenvalues and the deformation of the spectral mesh. Using this data, we approximate the stability domain in the space of system's parameters.

Confirming that the perturbation by conservative forces can yield flutter only in the supercritical region, we come to new qualitative conclusions. For $\delta \neq 0$ there exist indefinite damping matrices of such a structure that always causes the subcritical flutter even when the eigenvalue branches of the unperturbed gyroscopic system are well-separated due to changes in the stiffness matrix that break the rotational symmetry of the

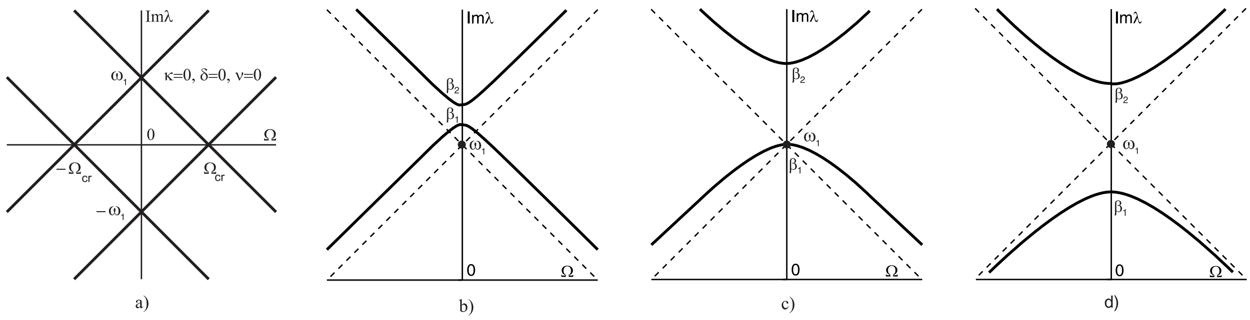

Figure 1: $2n = 2$: (a) spectral mesh and its conservative deformation ($\kappa > 0$) in the case when (b) $\mathbf{K}$ is positive-definite, (c) $\mathbf{K}$ is positive semi-definite, and (d) $\mathbf{K}$ is indefinite.

rotor. In the presence of indefinite damping the regions of flutter in the three-dimensional space of the parameters Ω, δ, and κ turn out to have a conical shape. The inclination of the cones is different in the sub- and supercritical ranges of Ω and it is substantially determined by the Krein signature of the eigenvalues involved in the crossings of eigenvalues corresponding to the apexes of the cones. Due to the different inclination the zones of the supercritical flutter are visible in the plane $\delta = 0$ whereas the intersection of this plane with the zones of subcritical flutter can be an empty set. A discovered singularity of the stability boundary allows for the combinations of dissipative and non-conservative positional forces yielding the subcritical flutter instability in the vicinity of the nodes of the spectral mesh even in the case, when the damping matrix is positive definite with some of its eigenvalues close to zero. The vanishing and negative eigenvalues of the damping matrix encourage the development of the subcritical flutter while zero eigenvalues of the matrix of non-conservative positional forces suppress it. The proposed approach provides guidance to a classification of dissipative and non-conservative perturbations by their ability to cause the subcritical flutter, which is helpful in checking and correcting particular models of disc brakes and other rotating elastic bodies of revolution having frictional contact.

2 Dissipation-induced subcritical flutter in the case of $2n = 2$ d.o.f.

Owing to its relative simplicity the case of two degrees of freedom $(n = 1)$ allows for the detailed stability analysis. Although the complete investigation of the spectral mesh and its deformation under both Hamiltonian and non-Hamiltonian perturbations in system (5) with arbitrary number of degrees of freedom would be very desirable for applications, a restriction to two dimensions is justified for demonstrating the basic ideas of our theory. On the other hand, two-dimensional models are widely employed in acoustics of friction [30], while our perturbative approach does not depend on the number of degrees of freedom.

For $n = 1$ the spectrum of the unperturbed operator $\mathbf{L}_0(\Omega)$ consists of four branches. In the subcritical region $|\Omega| < \Omega_{cr} = \omega_1$ they cross in the plane $(\Omega, \mathrm{Im}\lambda)$ at the points $(0, \pm\omega_1)$, see Fig. 1(a). Assuming without loss in generality $\mathbf{N} = \mathbf{J}$, we consider a general perturbation of the gyroscopic system $\mathbf{L}_0(\Omega) + \Delta\mathbf{L}(\Omega)$. The size of the perturbation $\Delta\mathbf{L}(\Omega) = \delta\lambda\mathbf{D} + \kappa\mathbf{K} + \nu\mathbf{N} \sim \varepsilon$ is small, where $\varepsilon = \|\Delta\mathbf{L}(0)\|$ is the Frobenius norm of the perturbation at $\Omega = 0$. For small Ω and ε perturbation of the double semi-simple eigenvalue $\lambda = i\omega_1$ with two orthogonal eigenvectors $\mathbf{u}_1$ and $\mathbf{u}_2$

$$\mathbf{u}_1 = \frac{1}{\sqrt{2\omega_1}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \mathbf{u}_2 = \frac{1}{\sqrt{2\omega_1}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{8}$$

is described by the asymptotic formula [4]

$$\lambda_p^{\pm} = i\omega_1 + i\Omega\frac{f_{11} + f_{22}}{2} + i\frac{\epsilon_{11} + \epsilon_{22}}{2} \pm i\sqrt{\frac{(\Omega(f_{11} - f_{22}) + \epsilon_{11} - \epsilon_{22})^2}{4} + (\Omega f_{12} + \epsilon_{12})(\Omega f_{21} + \epsilon_{21})}, \tag{9}$$

where the quantities f_{jk} are

$$f_{jk} = \mathbf{u}_k^T \frac{\partial \mathbf{L}_0(\Omega)}{\partial \Omega}\mathbf{u}_j\bigg|_{\Omega=0,\lambda=i\omega_1} = 2i\omega_1 \mathbf{u}_k^T \mathbf{G}\mathbf{u}_j. \tag{10}$$

Due to the property $\mathbf{G} = -\mathbf{G}^T$ with the vectors (8) the formula (10) yields $f_{11} = f_{22} = 0$ and $f_{12} = -f_{21} = i$. The coefficients ϵ_{jk} are small complex numbers of order ε

$$\epsilon_{jk} = \mathbf{u}_k^T \Delta\mathbf{L}(0)\mathbf{u}_j = i\omega_1\delta\mathbf{u}_k^T\mathbf{D}\mathbf{u}_j + \kappa\mathbf{u}_k^T\mathbf{K}\mathbf{u}_j + \nu\mathbf{u}_k^T\mathbf{N}\mathbf{u}_j. \tag{11}$$

With the vectors (8) we obtain

$$\mathrm{Re}\lambda = -\frac{\mu_1+\mu_2}{4}\delta \pm \sqrt{\frac{|c|+\mathrm{Re}c}{2}}, \qquad \mathrm{Im}\lambda = \omega_1 + \frac{\rho_1+\rho_2}{4\omega_1}\kappa \pm \sqrt{\frac{|c|-\mathrm{Re}c}{2}}, \tag{12}$$

$$\mathrm{Re}c = \left(\frac{\mu_1-\mu_2}{4}\right)^2\delta^2 - \left(\frac{\rho_1-\rho_2}{4\omega_1}\right)^2\kappa^2 - \Omega^2 + \frac{\nu^2}{4\omega_1^2}, \qquad \mathrm{Im}c = \frac{\Omega\nu}{\omega_1} - \delta\kappa\frac{2\mathrm{tr}\mathbf{KD} - \mathrm{tr}\mathbf{K}\mathrm{tr}\mathbf{D}}{8\omega_1}, \tag{13}$$

where the eigenvalues $\mu_{1,2}$ and $\rho_{1,2}$ of the matrices $\mathbf{D}$ and $\mathbf{K}$ satisfy the equations

$$\mu^2 - \mu\mathrm{tr}\mathbf{D} + \det\mathbf{D} = 0, \qquad \rho^2 - \rho\mathrm{tr}\mathbf{K} + \det\mathbf{K} = 0. \tag{14}$$

Formulas (12) take into account the forces of all types and explicitly describe the perturbed spectrum by means of the eigenelements and the derivatives of the operator with respect to parameters, calculated solely at the nodes of the spectral mesh. This is more efficient for describing the veering and merging of eigenvalue branches [44], than the sensitivity analysis of simple eigenvalues of the works [5], [45].

2.1 Conservative deformation of the spectral mesh

Since the eigenvalues at the crossings in the subcritical range have the same Krein signature, they veer away under potential perturbation $\kappa\mathbf{K}$, destroying the rotational symmetry of the body, as shown in Fig. 1(b)-(c). The conservative perturbation does not shift the eigenvalues from the imaginary axis, preserving the marginal stability. From expressions (12) and (13) we find that near the node $(0,\omega_1)$ in the plane $(\Omega, \mathrm{Im}\lambda)$

$$\left(\mathrm{Im}\lambda - \omega_1 - \frac{\rho_1+\rho_2}{4\omega_1}\kappa\right)^2 - \Omega^2 = \left(\frac{\rho_1-\rho_2}{4\omega_1}\right)^2\kappa^2, \quad \mathrm{Re}\lambda = 0. \tag{15}$$

For $\kappa \neq 0$, equation (15) describes a hyperbola with the asymptotes

$$\mathrm{Im}\lambda = \omega_1 + \frac{\rho_1+\rho_2}{4\omega_1}\kappa \pm \Omega. \tag{16}$$

The asymptotes cross each other above the node $(0,\omega_1)$ of the non-deformed spectral mesh for $\mathrm{tr}\mathbf{K} > 0$, exactly at the node for $\rho_1 = -\rho_2$, and below the node for $\mathrm{tr}\mathbf{K} < 0$. The branches of the hyperbola intersect the axis $\Omega = 0$ at the points

$$\beta_1 = \omega_1 + \frac{\rho_1}{2\omega_1}\kappa, \quad \beta_2 = \omega_1 + \frac{\rho_2}{2\omega_1}\kappa. \tag{17}$$

If the eigenvalues $\rho_{1,2}$ have the same sign, the intersection points are above the node for $\mathbf{K} > 0$ and below it for $\mathbf{K} < 0$, see Fig. 1(b). When one of the eigenvalues $\rho_{1,2}$ is zero, which implies semi-definiteness of the matrix $\mathbf{K}$, one of the branches of the hyperbola passes through the node. The other one crosses the axis $\Omega = 0$ above the node, if $\mathbf{K} \geq 0$ or below it, if $\mathbf{K} \leq 0$, Fig. 1(c). If $\mathbf{K}$ is indefinite, one of the points $\beta_{1,2}$ is located above the node and another one below it, Fig. 1(d).

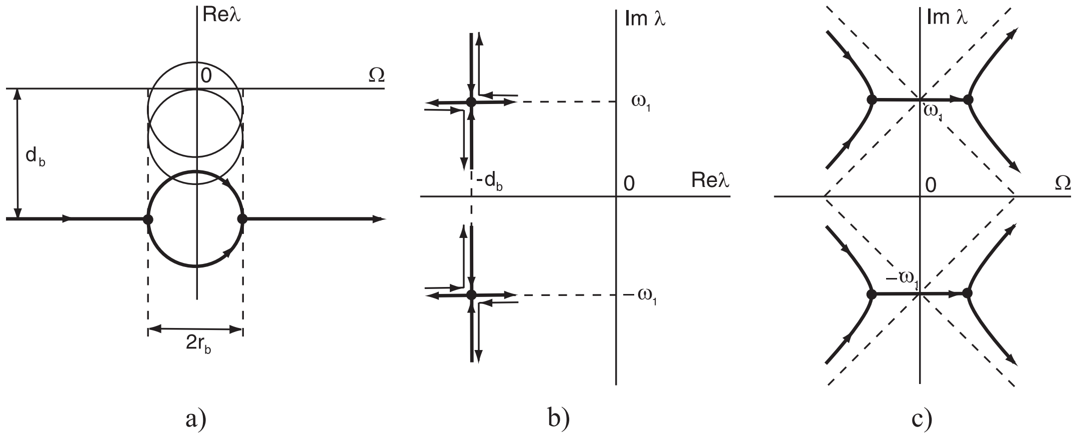

Figure 2: Origination of a latent source of subcritical flutter instability in presence of full dissipation: Submerged bubble of instability (a); coalescence of eigenvalues in the complex plane at two exceptional points (b); hyperbolic trajectories of imaginary parts (c).

2.2 Creating and activating the latent sources of instability by dissipation

Assuming $\nu = \kappa = 0$ in expressions (12) and (13) we find that

$$\left(\mathrm{Re}\lambda + \frac{\mu_1+\mu_2}{4}\delta\right)^2 + \Omega^2 = \frac{(\mu_1-\mu_2)^2}{16}\delta^2, \quad \mathrm{Im}\lambda = \omega_1 \quad for \quad \mathrm{Re}c > 0, \tag{18}$$

$$\Omega^2 - (\mathrm{Im}\lambda - \omega_1)^2 = \frac{(\mu_1-\mu_2)^2}{16}\delta^2, \quad \mathrm{Re}\lambda = -\frac{\mu_1+\mu_2}{4}\delta \quad for \quad \mathrm{Re}c < 0. \tag{19}$$

In the three-dimensional space $(\Omega, \mathrm{Im}\lambda, \mathrm{Re}\lambda)$ the circle of complex eigenvalues (18) belongs to the plane $\mathrm{Im}\lambda = \omega_1$, while the hyperbola (19) lies in the plane $\mathrm{Re}\lambda = -\delta(\mu_1+\mu_2)/4$, as shown in Fig. 2(a,c).

According to (18) the radius of the bubble of instability r_b and the distance d_b of its center from the plane $\mathrm{Re}\lambda = 0$ are defined by the eigenvalues $\mu_{1,2}$ of $\mathbf{D}$

$$r_b = |(\mu_1-\mu_2)\delta|/4, \quad d_b = |(\mu_1+\mu_2)\delta|/4. \tag{20}$$

The bubble of complex eigenvalues and hence the branches of the adjacent hyperbola (19) are "submerged" under the surface $\mathrm{Re}\lambda = 0$, when the conditions $d_b \geq r_b$ and $\delta\mathrm{tr}\mathbf{D} > 0$ are fulfilled, yielding the positive (semi-)definite matrix $\delta\mathbf{D}$ of (pervasive) full damping. In the complex plane the eigenvalues move with the variation of Ω along the lines $\mathrm{Re}\lambda = -d_b$ until they meet at the junction of the bubble of instability (18) with the hyperbola (19)

$$\mathrm{Im}\lambda = \omega_1, \quad \mathrm{Re}\lambda = -\delta(\mu_1+\mu_2)/4, \quad \Omega = \pm\delta(\mu_1-\mu_2)/4 \tag{21}$$

and form the double eigenvalue with the Jordan chain of two generalized eigenvectors. With the further increase in Ω the eigenvalues split in the orthogonal direction, never crossing the imaginary axis, Fig. 2(b).

For the phenomenon of squeal it is important that the dissipation-induced bubble of complex eigenvalues, localized in the subcritical interval $|\Omega| < \Omega_d$, is a latent source of unstable modes with the frequencies close to the repeated eigenfrequency $\mathrm{Im}\lambda = \omega_1$ of the non-rotating system. In the absence of circulatory forces the radius of the bubble of instability (18) is greater than the depth of its submersion under the surface $\mathrm{Re}\lambda = 0$, only if the eigenvalues $\mu_{1,2}$ of $\mathbf{D}$ have different signs. The eigenvalues of the emerged bubble have positive real parts in the range $\Omega^2 < \Omega_{cr}^2$, where $\Omega_{cr} = \frac{\delta}{2}\sqrt{-\det\mathbf{D}}$, confirming that the negative friction-velocity gradient as a source of indefinite damping can be a reason for subcritical flutter and squeal.

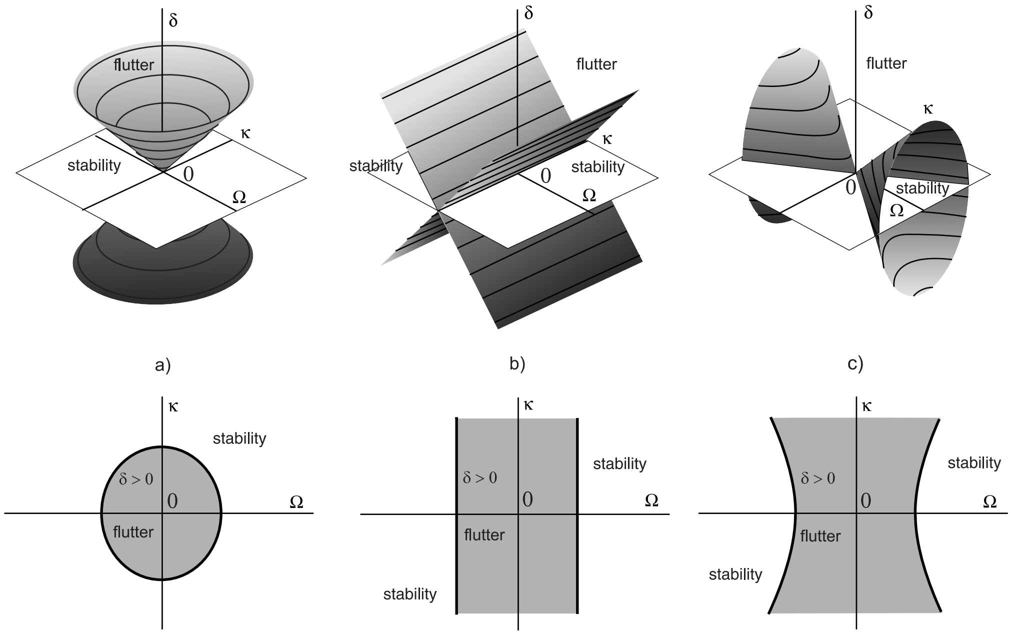

Figure 3: n=2: domains of the subcritical flutter instability (parametric resonance) in the absence of the non-conservative positional forces ($\nu = 0$) for the idefinite matrix $\mathbf{D}$ with $\mathrm{tr}\mathbf{D} > 0$, $\det \mathbf{D} < 0$, and (a) $A > 0$, (b) $A = 0$, (c) $A < 0$.

The sector-shaped domain of asymptotic stability of system (1) with indefinite damping is defined by the constraints $\delta \mathrm{tr}\mathbf{D} > 0$ and $\Omega^2 > \Omega_{cr}^2$. Due to the singularity at the origin in the plane (δ, Ω), an unstable system with indefinite damping can be stabilized by sufficiently strong gyroscopic forces. With the increase in $\det \mathbf{D}$ the stability domain gets wider and for $\det \mathbf{D} > 0$ it is defined by the condition $\delta \mathrm{tr}\mathbf{D} > 0$. At $\det \mathbf{D} = 0$ the line $\Omega = 0$ does not belong to the domain of asymptotic stability. Changing the matrix $\delta\mathbf{D}$ from positive definite to indefinite triggers the state of the bubble of instability from the latent $(\mathrm{Re}\lambda < 0)$ to the active one $(\mathrm{Re}\lambda > 0)$, see Fig. 2(a).

2.3 Conical zones of the subcritical flutter induced by the indefinite damping

Now we show that even if the eigenvalues of the rotationally symmetric gyroscopic system (1) are separated in the subritical region by the symmetry-breaking variation of the stiffness matrix $\kappa\mathbf{K}$, the inclusion of dissipation $\delta\mathbf{D}$ with the indefinite matrix $\mathbf{D}$ can cause flutter instability. Indeed, with $\nu = 0$ in equation (12), the condition $\mathrm{Re}\lambda < 0$ yields the linear approximation to the domain of asymptotic stability in the space of the parameters δ, Ω, and κ

$$\delta \mathrm{tr}\mathbf{D} > 0, \quad \kappa^2 A + \Omega^2 (2\omega_1 \mathrm{tr}\mathbf{D})^2 > -\det \mathbf{D} (\omega_1 \mathrm{tr}\mathbf{D})^2 \delta^2. \tag{22}$$

For the damping matrices $\mathbf{D} > 0$ the conditions (22) are always fulfilled, whereas for the indefinite damping matrices with $\det \mathbf{D} < 0$ the expressions follow from (22) for the flutter instability domain, which has a form of the half of a cone for $A := \det \mathbf{D}(\rho_1 - \rho_2)^2 + (k_{12}(d_{22} - d_{11}) - d_{12}(k_{22} - k_{11}))^2 > 0$, the dihedral angle for $A = 0$, and the domain adjacent to a half of a cone for $A < 0$, see Fig. 3(a)-(c). The orientation of the instability domain is determined also by the Krein signature of the eigenvalues involved into the corresponding crossing, which is substantially different in the subcritical and in the supercritical regions. In the plane (Ω, κ) for a fixed $\delta > 0$ the instability domain has, respectively, the form of an ellipse, a stripe, or a region contained between the branches of a hyperbola. The latter case shows that a widely known in the engineering practice approach to the squeal suppression by reducing the rotational symmetry of the rotor is

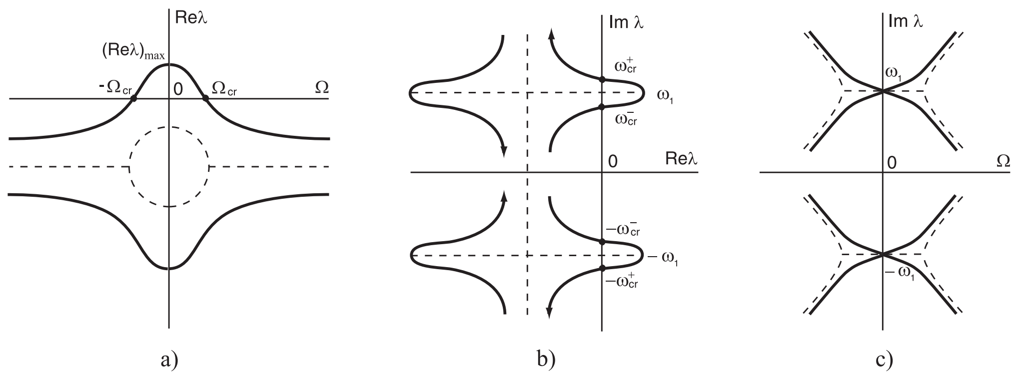

Figure 4: Subcritical flutter caused by the dissipative and circulatory forces: Collapse and emersion of the bubble of instability (a); excursions of eigenvalues to the right-hand side of the complex plane when Ω increases (b); crossing of the imaginary parts (c).

not efficient in the presence of indefinite damping, which originates from the brake pads with the negative friction-velocity gradient [12, 21]. Note that the threshold $A = 0$ separating the indefinite damping matrices was found first in [26] for a general two-dimensional non-conservative gyroscopc system with dissipation.

2.4 Activating the bubble of instability by non-conservative positional forces

In the absence of dissipation, the non-conservative positional forces destroy the marginal stability of gyroscopic systems [46]. Assuming $\delta = \kappa = 0$ in (12) and (13) we find that the eigenvalues of the branches $\pm(i\omega_1 + i\Omega)$ of the spectral mesh get positive real parts due to a non-conservative perturbation

$$\lambda_p^{\pm} = i\omega_1 \pm i\Omega \pm \frac{\nu}{2\omega_1}, \quad \lambda_n^{\pm} = -i\omega_1 \pm i\Omega \mp \frac{\nu}{2\omega_1}. \tag{23}$$

In contrast to the effect of indefinite damping, the circulatory forces destabilize one of the two modes at every Ω, Fig. 3(b). In order to localize the instability in the vicinity of the nodes, a combination of circulatory and dissipative forces is required.

With $\kappa = 0$ in (12) and (13) we describe the trajectories of the eigenvalues in the complex plane in presence of dissipative and non-conservative perturbations

$$\left(\mathrm{Re}\lambda + \frac{\mathrm{tr}\mathbf{D}}{4}\delta\right)(\mathrm{Im}\lambda - \omega_1) = \frac{\Omega\nu}{2\omega_1}. \tag{24}$$

Circulatory forces destroy the merging of modes shown in Fig. 2, causing the eigenvalues to move along the separated trajectories. According to (23) and (24) the eigenvalues with $|\mathrm{Im}\lambda|$ increasing with an increase in $|\Omega|$ move closer to the imaginary axis than the others, as shown in Fig 4(b). The non-conservative perturbation separates the bubble of instability and the adjacent hyperbolic eigenvalue branches into two non-intersecting curves in the space $(\Omega, \mathrm{Im}\lambda, \mathrm{Re}\lambda)$. The remnants of the original bubble of instability yield subcritical flutter at a frequency $\omega_{cr}^- < \omega < \omega_{cr}^+$ with the gyroscopic parameter in the range $\Omega^2 < \Omega_{cr}^2$, where

$$\Omega_{cr} = \delta\frac{\mathrm{tr}\mathbf{D}}{4}\sqrt{-\frac{\nu^2 - \delta^2\omega_1^2 \det \mathbf{D}}{\nu^2 - \delta^2\omega_1^2(\mathrm{tr}\mathbf{D}/2)^2}}, \quad \omega_{cr}^{\pm} = \omega_1 \pm \frac{\nu}{2\omega_1}\sqrt{-\frac{\nu^2 - \delta^2\omega_1^2 \det \mathbf{D}}{\nu^2 - \delta^2\omega_1^2(\mathrm{tr}\mathbf{D}/2)^2}}. \tag{25}$$

Hence, in the presence of the non-conservative positional forces the excursions of eigenvalues to the right-hand side of the complex plane shown in Fig. 4(b) are possible, even if the dissipation is full $(\det \mathbf{D} > 0)$.

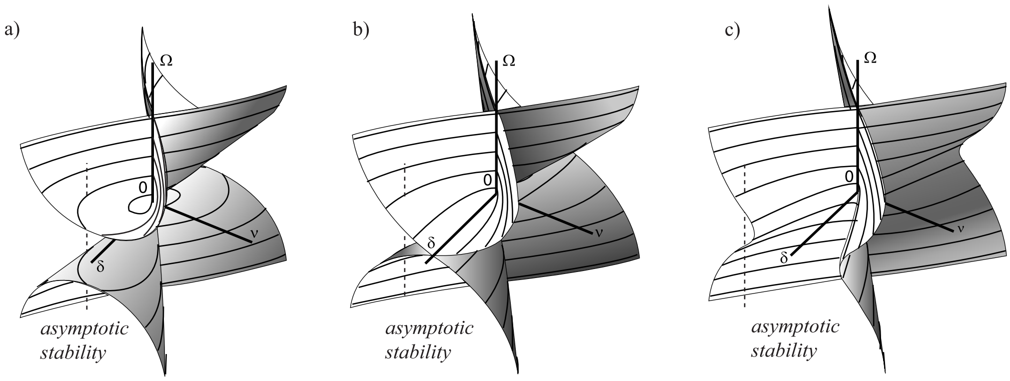

Figure 5: Domains of asymptotic stability in the space (δ, ν, Ω) for different types of damping: $\det \mathbf{D} < 0$ (a), $\det \mathbf{D} = 0$ (b), $\det \mathbf{D} > 0$ (c).

Extracting ν in the first of equations (25) yields approximation to the stability boundary in the space of the parameters δ, ν, and Ω

$$\nu = \pm\delta\omega_1 \mathrm{tr}\mathbf{D}\sqrt{\frac{\delta^2 \det \mathbf{D} + 4\Omega^2}{\delta^2(\mathrm{tr}\mathbf{D})^2 + 16\Omega^2}}. \tag{26}$$

If $\det \mathbf{D} \geq 0$ and Ω is fixed, the formula (26) describes two independent curves in the plane (δ, ν), intersecting with each other at the origin along the straight lines $2\nu = \pm\omega_1 \mathrm{tr}\mathbf{D}\delta$. For $\det \mathbf{D} < 0$ equation (26) describes in the plane (δ, ν) two branches of a closed loop, self-intersecting at the origin with the tangents $2\nu = \pm\omega_1 \mathrm{tr}\mathbf{D}\delta$. In the space of the three parameters the surface (26) is a cone with the "8"–shaped loop in a cross-section, see Fig. 5(a). Asymptotic stability is inside the two of four pockets of the cone, selected by the inequality $\delta \mathrm{tr}\mathbf{D} > 0$. The singularity at the origin is the degeneration of a more general configuration found in [26].

The domain of asymptotic stability bifurcates with the change of sign of $\det \mathbf{D}$. In case of indefinite damping an instability gap exists due to the singularity at the origin, Fig. 5(a). For $\det \mathbf{D} = 0$ the gap vanishes in the direction $\nu = 0$, Fig. 5(b). Despite the full dissipation with $\det \mathbf{D} > 0$ unfolds the singularity, the memory about the instability gap is preserved in the two folds of the stability boundary with the locally strong curvature, Fig. 5(c). When both $\mu_1 > 0$ and $\mu_2 > 0$, the folds are more pronounced, if one of the eigenvalues is close to zero. If the eigenvalues $\mu_{1,2}$ have different signs, subcritical flutter is possible for any combination of δ and ν including the case when the non-conservative positional forces are absent $(\nu = 0)$.

Independently on the structure of the matrix $\mathbf{D}$, the primary role of dissipation is the creation of the bubble of instability. It is submerged below the surface $\mathrm{Re}\lambda = 0$ in the space $(\Omega, \mathrm{Im}\lambda, \mathrm{Re}\lambda)$ in case of full dissipation and partially lies in the domain $\mathrm{Re}\lambda > 0$ when damping is indefinite. Non-conservative positional forces destroy the bubble into two branches and shift one of them to the region of positive real parts even in case of full dissipation. Since the branch remembers the existence of the bubble, the subcritical flutter is developing near the nodes of the spectral mesh.

3 Example: A rotating circular string

The perturbative approach of the previous section, modified along the lines of the work [3], is applicable to the *non-discretized* boundary eigenvalue problems, associated with the rotating strings, rings, discs, and shells in frictional contact for a wide class of available boundary conditions. We notice, however, that the correct formulation of the boundary conditions for such problems is a delicate question, which is not resolved yet in full in the existing literature, see, e.g., [36].

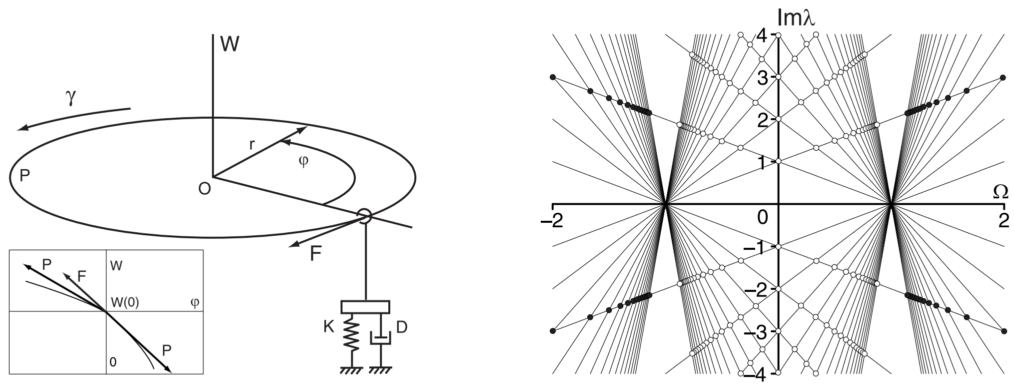

Figure 6: A rotating circular string and its “keyboard” constituted by the nodes (marked by white and black) of the spectral mesh (only 30 modes are shown).

The eigenvalue behavior predicted by the analysis of the general two-dimensional system of the previous section was already observed in the works [5, 11, 40], who studied a rotating disc and a rotating circular string in a point-wise contact with the stationary load systems.

For simplicity, following [5] we consider a circular string of displacement $W(\varphi,\tau)$, radius r, and mass per unit length ρ that rotates with the speed γ and passes at $\varphi=0$ through a massless eyelet generating a constant frictional follower force F on the string, as shown in Fig. 6. The circumferential tension P in the string is assumed to be constant; the stiffness of the spring supporting the eyelet is K and the damping coefficient of the viscous damper is D; the velocity of the string in the φ direction has constant value γr. This a somewhat artificial system contains, however, the fundamental physics of interest, i.e. the interaction of rotating flexible medium with a stationary constraint in which the inertial, gyroscopic, and centripetal acceleration effects, together with the stiffness effects of the medium, are in dynamic equilibrium with the forces generated by the constraint. With the non-dimensional variables and parameters

$$t=\frac{\tau}{r}\sqrt{\frac{P}{\rho}},\quad w=\frac{W}{r},\quad \Omega=\gamma r\sqrt{\frac{\rho}{P}},\quad k=\frac{Kr}{P},\quad \mu=\frac{F}{P},\quad d=\frac{D}{\sqrt{\rho P}} \tag{27}$$

the substitution of $w(\varphi,t)=u(\varphi)\exp(\lambda t)$ into the governing equation and boundary conditions yields the boundary eigenvalue problem [5]

$$Lu=\lambda^2 u+2\Omega\lambda u'-(1-\Omega^2)u''=0, \tag{28}$$

$$u(0)-u(2\pi)=0,\quad u'(0)-u'(2\pi)=\frac{\lambda d+k}{1-\Omega^2}u(0)+\frac{\mu}{1-\Omega^2}u'(0), \tag{29}$$

where $'=\partial_\varphi$. The non-self-adjoint boundary eigenvalue problem (28) and (29) depends on the speed of rotation (Ω), and damping (d), stiffness (k), and friction (μ) coefficients of the constraint.

Since the unconstrained rotating string is a gyroscopic system, the eigenfunctions of the adjoint eigenvalue problems, corresponding to a purely imaginary eigenvalue λ, coincide. With $u=C_1\exp\left(\varphi\lambda/(1-\Omega)\right)+C_2\exp\left(-\varphi\lambda/(1+\Omega)\right)$ assumed as a solution of (28) in (29), we find the characteristic equation, whose roots yield the eigenvalues of the eigenvalue problem (28), (29)

$$\lambda_n^+=in(1+\Omega),\quad \lambda_n^-=in(1-\Omega),\quad n\in\mathbb{Z}, \tag{30}$$

with the eigenfunctions $u_n^\pm=\cos(n\varphi)\mp i\sin(n\varphi)$. Two eigenvalue branches $\lambda_n^\varepsilon=in(1+\varepsilon\Omega)$ and $\lambda_m^\delta=im(1+\delta\Omega)$, where $\varepsilon,\delta=\pm1$, intersect each other at the node $(\Omega_{mn}^{\varepsilon\delta},\lambda_{mn}^{\varepsilon\delta})$ with

$$\Omega_{mn}^{\varepsilon\delta}=\frac{n-m}{m\delta-n\varepsilon},\quad \lambda_{mn}^{\varepsilon\delta}=\frac{inm(\delta-\varepsilon)}{m\delta-n\varepsilon}, \tag{31}$$

where the double eigenvalue $\lambda_{mn}^{\varepsilon\delta}$ has two linearly independent eigenfunctions

$$u_n^\varepsilon = \cos(n\varphi) - \varepsilon i \sin(n\varphi), \quad u_m^\delta = \cos(m\varphi) - \delta i \sin(m\varphi). \tag{32}$$

Intersections (31), corresponding to the forward and backward traveling waves, occur in the subcritical region ($|\Omega| < 1$) and are marked in Fig. 6 by white dots. The black dots indicate the intersections of the forward and reflected waves taking place in the supercritical region ($|\Omega| > 1$).

Using the perturbation theory [3, 8, 43] and taking into account expressions (31) and (32) we find an expression for the eigenvalues originated after the splitting of the double eigenvalues due to interaction of the rotating string with the external loading system

$$\lambda = \lambda_{nm}^{\varepsilon\delta} + i\frac{\varepsilon n + \delta m}{2}\Delta\Omega + i\frac{n+m}{8\pi nm}(d\lambda_{nm}^{\varepsilon\delta} + k) + \frac{\varepsilon+\delta}{8\pi}\mu \pm \sqrt{c}, \tag{33}$$

where $\Delta\Omega = \Omega - \Omega_{nm}^{\varepsilon\delta}$, and

$$c = \left(i\frac{\varepsilon n - \delta m}{2}\Delta\Omega + i\frac{m-n}{8\pi mn}(d\lambda_{nm}^{\varepsilon\delta} + k) + \frac{\varepsilon - \delta}{8\pi}\mu\right)^2 - \frac{(d\lambda_{nm}^{\varepsilon\delta} + k - i\varepsilon n\mu)(d\lambda_{nm}^{\varepsilon\delta} + k - i\delta m\mu)}{16\pi^2 nm}.$$

Due to action of gyroscopic forces and an external spring double eigenvalues $\lambda_{nm}^{\varepsilon\delta}$ split in the subcritical region $|\Omega| < 1$ ($\varepsilon < 0$, $\delta > 0$ and $m > n > 0$) as

$$\lambda = \lambda_{nm}^{\varepsilon\delta} + i\frac{m-n}{2}\Delta\Omega + i\frac{n+m}{8\pi nm}k \pm i\sqrt{\frac{k^2}{16\pi^2 nm} + \left(\frac{m-n}{8\pi mn}k - \frac{m+n}{2}\Delta\Omega\right)^2}, \tag{34}$$

while in the supercritical region $|\Omega| > 1$ ($\varepsilon < 0$, $\delta > 0$ and $m > 0$, $n < 0$)

$$\lambda = \lambda_{nm}^{\varepsilon\delta} + i\frac{m+|n|}{2}\Delta\Omega + i\frac{|n|-m}{8\pi|n|m}k \pm \sqrt{\frac{k^2}{16\pi^2|n|m} - \left(\frac{|n|-m}{2}\Delta\Omega - \frac{m+|n|}{8\pi m|n|}k\right)^2}. \tag{35}$$

Therefore, for $|\Omega| < 1$ the spectral mesh collapses into separated curves demonstrating avoided crossings; for $|\Omega| > 1$ the eigenvalue branches overlap forming the bubbles of instability with eigenvalues having positive real parts, see Fig. (7)(a). From (35) a linear approximation follows to the boundary of the domains of supercritical flutter instability in the plane (Ω, k) (gray resonance tongues in Fig. (7)(b))

$$k = \frac{4\pi|n|m(|n|-m)}{(\sqrt{|n|} \pm \sqrt{m})^2}\left(\Omega - \frac{|n|+m}{|n|-m}\right). \tag{36}$$

In the subcritical region we focus on the nodes of the spectral mesh at $\Omega = 0$ as the most relevant to the problems of acoustics of friction. Since in this case $m = n$ and $\varepsilon = -\delta$, we find that the double eigenvalue in splits due to action of gyroscopic forces and an external spring as

$$\lambda = in + i\frac{k}{4\pi n} \pm i\sqrt{n^2\Omega^2 + \frac{k^2}{16\pi^2 n^2}} \tag{37}$$

demonstrating the avoided crossing, see Fig. (7)(a). The effect of damping and gyroscopic forces yields

$$\left(\mathrm{Re}\lambda + \frac{d}{4\pi}\right)^2 + n^2\Omega^2 = \frac{d^2}{16\pi^2}, \quad \mathrm{Im}\lambda = n, \tag{38}$$

$$n^2\Omega^2 - (\mathrm{Im}\lambda - n)^2 = \frac{d^2}{16\pi^2}, \quad \mathrm{Re}\lambda = -\frac{d}{4\pi}, \tag{39}$$

The lower branch of the hyperbola (37) passes through the node $\mathrm{Im}\lambda = n$, while the upper one intersects the axis $\Omega = 0$ at $\mathrm{Im}\lambda = n + \frac{k}{2\pi n}$ in the plane $(\Omega, \mathrm{Im}\lambda)$, see Fig. 7(a). In the two-dimensional case the reason

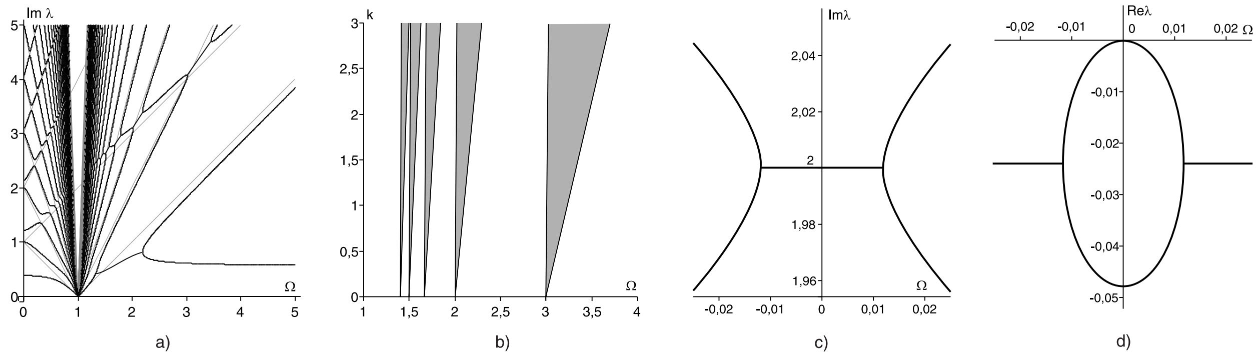

Figure 7: (a) Deformation of the spectral mesh of the rotating string interacting with the external spring with $k = 2$, (b) approximation (36) to the corresponding tongues of the supercritical flutter, (c,d) effect of the external damper with $d = 0.3$ near the node $(0, 2)$ of the spectral mesh.

for such a degenerate behavior is zero eigenvalue in the matrix $\mathbf{K}$ of external potential fores. The external damper creates a latent source of subcritical flutter instability exactly as it happens in two dimensions when $\mathbf{D}$ has one zero eigenvalue. Indeed, the bubble of instability (38) together with the adjacent hyperbola (39) is under the plane $\mathrm{Re}\lambda = 0$, touching it at the origin, as shown in Fig. 7(c),(d).

Deformation patterns of the spectral mesh obtained by the perturbation theory and shown in Fig. 7, qualitatively agree with the results of numerical calculations for the string [5] and for the disc [40]. They show that the perturbations from a *point-wise* external source of potential, damping, and friction forces are degenerate. Even without the friction term in (29) the degeneracy of the model persists, as is clearly seen from the comparison of Fig. 7(a),(d) with Fig. 1(c) and Fig. 2(a). Similar effect was detected for the rotating disc in a point-wise frictional contact in [11, 40].

Below we show that the degeneracy of the perturbation can be easily resolved. For this purpose we consider a discretized equations of the string, which follow from (1) with $\omega_s = s$. We find the structure of the matrices $\mathbf{D}$ and $\mathbf{K}$ from the desired behaviour of the eigenvalues originated after the splitting of the double eigenvalues at the nodes of the spectral mesh.

Assuming for simplicity that $n = 2$ and, hence, $\omega_1 = 1$, $\omega_2 = 2$, we find that the eigenvalue branches of the spectral mesh cross in the supercritical region $|\Omega| > 1$ of the $(\Omega, \mathrm{Im}\lambda)$-plane at the four points $(\pm 3, \pm 4)$. In the subcritical region $|\Omega| < 1$ there exist eight crossings $(\pm 1/3, \pm 4/3)$, $(0, \pm 1)$, and $(0, \pm 2)$, see Fig. 8(a). Splitting of the double eigenvalue $\lambda_0 = i\omega_0$ with the eigenvalues $\mathbf{u}_1$ and $\mathbf{u}_2$ at $\Omega = \Omega_0$ is given by the formula $\det(\mathbf{F} + (\lambda - \lambda_0)\mathbf{G}) = 0$, where the entries of the matrices $\mathbf{G}$ and $\mathbf{F}$ are [3]

$$\begin{aligned} G_{ij} &= 2i\omega_0 \bar{\mathbf{u}}_i^T \mathbf{u}_j + 2\Omega_0 \bar{\mathbf{u}}_i^T \mathbf{G}\mathbf{u}_j, \\ F_{ij} &= (2i\omega_0 \bar{\mathbf{u}}_i^T \mathbf{G}\mathbf{u}_j + 2\Omega_0 \bar{\mathbf{u}}_i^T \mathbf{G}^2 \mathbf{u}_j)(\Omega - \Omega_0) + i\omega_0 \bar{\mathbf{u}}_i^T \mathbf{D}\mathbf{u}_j \delta + \bar{\mathbf{u}}_i^T \mathbf{K}\mathbf{u}_j \kappa + \bar{\mathbf{u}}_i^T \mathbf{N}\mathbf{u}_j \nu. \end{aligned} \quad (40)$$

For example, at the supercritical intersection of the branches $i(1 + \Omega)$ and $-2i(1 - \Omega)$ the double eigenvalue $\lambda_0 = 4i$, at $\Omega_0 = 3$ has the eigenvectors $\mathbf{u}_1 = (-i/2, 1/2, 0, 0)^T$, $\quad \mathbf{u}_2 = \sqrt{2}(0, 0, -i/4, 1/4)^T$. Calculating the matrices $\mathbf{G}$ and $\mathbf{K}$ and then the first-order approximations to the eigenvalues, we find that the perturbation of the stiffness matrix $\kappa\mathbf{K}$ yields the supercritical flutter when

$$\kappa > \frac{8(\Omega - 3)}{\mathrm{tr}\mathbf{K} + k_{11} + k_{22} \pm \sqrt{8(k_{14} - k_{23})^2 + 8(k_{13} + k_{24})^2}}, \quad (41)$$

where k_{ij} are the entries of the matrix $\mathbf{K}$. The subcritical intersection of the modes $2i(1 - \Omega)$ and $i(1 + \Omega)$ at $\Omega_0 = 1/3$ originates the double eigenvalue $\lambda_0 = 4i/3$ with the eigenvectors $\mathbf{u}_1 = (-i/2, 1/2, 0, 0)^T$ and $\mathbf{u}_2 = \sqrt{2}(0, 0, i/4, 1/4)^T$. It is easy to verify that in this case the instability domain in the Ω, κ-plane does

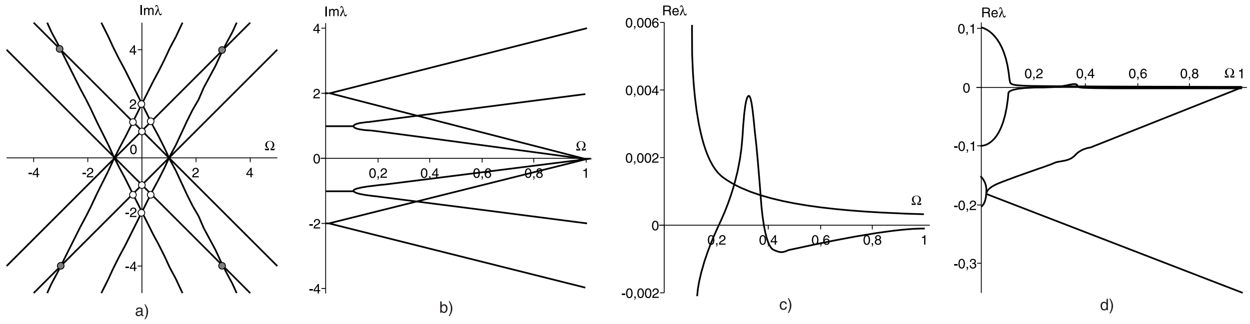

Figure 8: A $2n = 4$-dimensional discretized model of the rotating circular string: (a) spectral mesh, (b) the effect of indefinite damping with $d = 0.1$ and stiffness variation with $\kappa = 0.003$ on (b) imaginary parts of the eigenvalues (c),(d) on the real parts.

not exist when $\delta = 0$ and $\nu = 0$. In the presence of damping subcritical flutter is possible, which follows from the formula for the eigenvalue splitting for the fixed $\Omega = \Omega_0 = 1/3$

$$\lambda=\frac{4i}{3}-\frac{4\delta\mathrm{Re}a-3i\kappa\mathrm{Im}a}{48}\pm\frac{1}{48}\sqrt{(4\delta\mathrm{Re}b-3i\kappa\mathrm{Im}b)^2+8(4\delta\mathrm{Re}c-3i\kappa\mathrm{Im}c)^2+8(4\delta\mathrm{Re}d-3i\kappa\mathrm{Im}d)^2}, \quad (42)$$

where

$$\begin{aligned}
\mathrm{Re}a &= 2d_{22}+d_{44}+d_{33}+2d_{11}, \quad \mathrm{Im}a = 2k_{11}+2k_{22}+k_{44}+k_{33},\\
\mathrm{Re}b &= 2d_{22}-d_{44}+2d_{11}-d_{33}, \quad \mathrm{Im}b = 2k_{11}-k_{33}-k_{44}+2k_{22},\\
\mathrm{Re}c &= d_{14}+d_{23}, \quad \mathrm{Im}c = k_{14}+k_{23}, \quad \mathrm{Re}d = d_{24}-d_{13}, \quad \mathrm{Im}d = k_{13}-k_{24}.
\end{aligned} \quad (43)$$

According to (42) in the κ, δ-plane the instability domain is inside of the sector

$$\delta > \pm\frac{3\kappa}{4\mathrm{Re}a}\sqrt{-\frac{(\mathrm{Im}b\mathrm{Re}b+\mathrm{Im}c\mathrm{Re}c+\mathrm{Im}d\mathrm{Re}d)^2-(\mathrm{Im}b^2+\mathrm{Im}c^2+\mathrm{Im}d^2)\mathrm{Re}a^2}{\mathrm{Re}b^2+\mathrm{Re}c^2+\mathrm{Re}d^2-\mathrm{Re}a^2}}, \quad (44)$$

which is inclined in such a manner that in the plane Ω, κ the flutter domain has the form of an ellipse. In the Ω, δ, κ-space the instability domain has the conical shape. The space orientation of the cones in the sub- and subcritical domains is substantially different, which explains the invisibility of the flutter domains in the subcritical range and their simultaneous existence in the supercritical range for $\delta = 0$. Using the condition (44) we easly construct the perturbations, yielding flutter near all the nodes in the subcritical region

$$\mathbf{D} = \begin{pmatrix} -2 & 0 & 0 & 1\\ 0 & 2 & 0 & 0\\ 0 & 0 & 3 & 0\\ 1 & 0 & 0 & 4 \end{pmatrix}, \qquad \mathbf{K} = \begin{pmatrix} 1 & 1 & 3 & 4\\ 1 & 5 & 3 & 2\\ 3 & 3 & 3 & 7\\ 4 & 2 & 7 & 7 \end{pmatrix}, \quad (45)$$

as is seen in Fig. 8(b)-(d). The matrices $\mathbf{D}$ and $\mathbf{K}$ are indefinite with the eigenvalues 2, 3, 4.162277660, -2.162277660 and 4.011016866, 15.39075619, -0.9102103430, -2.491562713, respectively.

Conclusion

Supporting an attractive thesis by Chan et al. [39], “Flutter instabilities in brake systems occur primarily as a result of symmetry [breaking]; the frictional mechanism which has been the subject of much research over the past forty years is of secondary importance,” the sensitivity analysis of the present paper demonstrates how the nodes of the spectral mesh, situated in the subcritical range, may serve as the “keyboard” of a rotating elastic body of revolution. The frictional contact is a source of non-Hamiltonian and symmetry-breaking perturbations. In the vicinity of the "keys" of the "keyboard" damping creates eigenvalue bubbles, which are dangerous by the ability to get positive real parts in presence of non-conservative positional forces or even without them, if the damping is indefinite. The activated bubbles of instability cause subcritical flutter of a rotating structure, forcing it to vibrate at a frequency close to the double frequency of the node and at the angular velocity close to that of the node. An advantage of the sensitivity analysis of the spectral mesh to arbitrary perturbations is in selecting the generic behavior of eigenvalues and thus the generic perturbations yielding flutter or stability. For example, the observed degeneracy in the movement of eigenvalues of the rotating string and disc evidences that a point-wise contact leads to the semi-definite perturbation operators, which suppress generic instability mechanism behind the squeal. The effect seems to be similar to the so-called Herrmann-Smith paradox of a beam resting on a uniform Winkler elastic foundation and loaded by a follower force [47]. Therefore, more correct description of the frictional contact would take into account the finite dimensions of the pads as well as the dependence of their characteristics on material coordinates. The size of the friction pads and their placement with respect to the rotating body should select the particular node of the spectral mesh that produces an unstable complex eigenvalue [48, 49]. The selection rules as well as the optimal distribution of the stiffness, damping, and friction characteristics of the pads can be effectively investigated with the approach developed in the present paper.

Acknowledgements

The work has been supported by the research grant DFG HA 1060/43-1. The author expresses his gratitude to Professor P. Hagedorn, Technische Universität Darmstadt, for valuable discussions.

References

[1] W. Nagata, N.S. Namachchivaya, *Bifurcations in gyroscopic systems with an application to rotating shafts*, Proc. R. Soc. Lond. A, Vol. 454, (1998), pp. 543-585.

[2] R. Hryniv, P. Lancaster, *Stabilization of Gyroscopic Systems*, Z. Angew. Math. Mech. Vol. 81 No. 10, (2001), pp. 675-681.

[3] O.N. Kirillov, A.P. Seyranian, *Collapse of the Keldysh chains and stability of continuous non-conservative systems*, SIAM J. Appl. Math., Vol. 64, No. 4, (2004), pp. 1383-1407.

[4] O.N. Kirillov, A.A. Mailybaev, A.P. Seyranian, *Singularities of energy surfaces under non-Hermitian perturbations.* Dokl. Phys., Vol. 50 No. 11, (2005), pp. 577-582.

[5] L. Yang, S.G. Hutton, *Interactions between an idealized rotating string and stationary constraints.* J. Sound Vibr., Vol. 185, No. 1, (1995), pp. 139-154.

[6] C. Gordon, D. L. Webb, S. Wolpert, *One cannot hear the shape of a drum*, Bull. Amer. Math. Soc. (N.S.), Vol. 27, (1992), pp. 134-138.

[7] R. Southwell, *On the free transverse vibrations of a uniform circular disc clamped at its centre; and on the effect of rotation*, Proc. R. Soc. Lond., Vol. 101, (1921), pp. 133-153.

[8] U. Günther, O.N. Kirillov, *A Krein space related perturbation theory for MHD α^2-dynamos and resonant unfolding of diabolical points*, J. Phys. A: Math. Gen. Vol. 39, No. 32, (2006), pp. 10057-10076.

[9] M. Dellnitz, I. Melbourne, 1994 *Generic movement of eigenvalues for equivariant self-adjoint matrices*, J. of Comput. and Appl. Math., Vol. 55, No. 3, (1994), pp. 249-259.

[10] W.D. Iwan, K.J. Stahl, *The response of an elastic disc with a moving mass system*, Trans. ASME. J. Appl. Mech. Vol. 40, (1973), pp. 445–451.

[11] S.G. Hutton, S. Chonan, B.F. Lehmann, *Dynamic response of a guided circular saw*, J. Sound Vibr. Vol. 112, No. 3, (1987), pp. 527-539.

[12] J.E. Mottershead, *Vibration- and friction-induced instabilities in discs.* The Shock and Vibration Digest. Vol. 30, No. 1, (1998), pp. 14-31.

[13] C.O. Chang, J.J. Hwang, C.S. Chou, *Modal precession of a rotating hemispherical shell*, Int. J. Solids Structures. Vol. 33, No. 19, (1996), pp. 2739-2757.

[14] Y. Fukumoto, Y. Hattori, *Curvature instability of a vortex ring*, J. Fluid Mech. Vol. 526, (2005), pp. 77-115.

[15] R.S. MacKay, *Stability of equilibria of Hamiltonian systems*, in S. Sarkar, editor, *Nonlinear Phenomena and Chaos*, Adam Hilger, Bristol (1986), pp. 254-270,

[16] R.T. Spurr, *The ringing of wine glasses*, Wear, Vol. 4, (1961), pp. 150-153.

[17] T.D. Rossing, *Acoustics of the glass harmonica*, J. Acoust. Soc. Am., Vol. 95, No. 2, (1994), pp. 1106-1111.

[18] A. Akay, *Acoustics of friction*, J. Acoust. Soc. Am., Vol. 111, No. 4, (2002), pp. 1525-1548.

[19] N.M. Kinkaid, O.M. O'Reilly, P. Papadopoulos, *Automotive disc brake squeal*, J. Sound Vib., Vol. 267, (2003), pp. 105-166.

[20] H. Ouyang, W. Nack, Y. Yuan, F. Chen, *Numerical analysis of automotive disc brake squeal: a review*, Int. J. Veh. Noise and Vibr., Vol. 1, No. 3–4, (2005), pp. 207-231.

[21] H. Ouyang, *Parametric vibration induced by moving loads*, in: F. Chen, C.A. Tan, R.L. Quaglia, editors, *Disc brake squeal: mechanism, analysis, evaluation, and reduction/prevention*, SAE, Warrendale, USA (2006), pp. 49-78.

[22] T.J. Bridges, *A geometric formulation of the conservation of wave action and its implications for signature and the classification of instabilities*, Proc. R. Soc. Lond. A, Vol. 453, (1997), pp. 1365-1395.

[23] R.T. Spurr, *A theory of brake squeal*, Proc. Inst. Mech. Engrs (AD), Vol. 1, (1961), pp. 33-52.

[24] P. Hagedorn, *Neuere Entwicklungen in der technischen Schwingungslehre*, Z. angew. Math. Mech., Vol. 68, (1988), pp. 22-31.

[25] P. Freitas, M. Grinfeld, P.A. Knight, *Stability of finite-dimensional systems with indefinite damping*, Adv. Math. Sci. Appl., Vol. 7, No. 1, (1997), pp. 437–448.

[26] O.N. Kirillov, *Destabilization paradox due to breaking the Hamiltonian and reversible symmetry*, Int. J. of Non-Lin. Mech., Vol. 42, No. 1, (2007) pp. 71-87.

[27] M. Kröger, M. Neubauer, K. Popp, *Experimental investigation on the avoidance of self-excited vibrations*, Phil. Trans. R. Soc. A, Vol. 366, (2008), pp. 785-810.

[28] M.R. North, Disc brake squeal, in: *Braking of Road Vehicles*, Automobile Division of the Institution of Mechanical Engineers, Mechanical Engineers Publications Limited, London, England (1976), pp. 169-176.

[29] J.-J. Sinou, L. Jezequel, 2007 *Mode coupling instability in friction-induced vibrations and its dependency on system parameters including damping*, Eur. J. Mech. A., Vol. 26, (2007), pp. 106-122.

[30] U. von Wagner, D. Hochlenert, P. Hagedorn, *Minimal models for disk brake squeal*, J. Sound Vibr., Vol. 302, No. 3, (2007), pp. 527-539.

[31] N. Hoffmann, L. Gaul, *Effects of damping on mode-coupling instability in friction induced oscillations*, Z. angew. Math. Mech. Vol. 83, (2003), pp. 524-534.

[32] O.N. Kirillov, *A theory of the destabilization paradox in non-conservative systems*, Acta Mech. Vol. 174, No. 3-4, (2005), pp. 145-166.

[33] O.N. Kirillov, *Bifurcation of the roots of the characteristic polynomial and destabilization paradox in friction-induced oscillations*, Theor. Appl. Mech., Vol. 34, No. 2, (2007), pp. 87-109.

[34] R. Krechetnikov, J.E. Marsden, *On destabilizing effects of two fundamental non-conservative forces*, Physica D, Vol. 214, (2006), pp. 25-32.

[35] O.N. Kirillov, *Gyroscopic stabilization in the presence of nonconservative forces*, Dokl. Math., Vol. 76, No. 2, (2007), pp. 780-785.

[36] G. Spelsberg-Korspeter, O.N. Kirillov, P. Hagedorn, *Modeling and stability analysis of an axially moving beam with frictional contact*, Trans. ASME, J. Appl. Mech., Vol. 75, No. 3, (2008), 031001.

[37] F. Massi, O. Giannini, L. Baillet L. 2006 *Brake squeal as dynamic instability: an experimental investigation*, J. Acoust. Soc. Am., Vol. 120, No. 3, (2006), pp. 1388-1398.

[38] F. Massi, O. Giannini, *Effect of damping on the propensity of squeal instability: An experimental investigation.* J. Acoust. Soc. Am., Vol. 123, No. 4, (2008), pp. 2017-2023.

[39] S.N. Chan, J.E. Mottershead, M.P. Cartmell, *Parametric resonances at subcritical speeds in discs with rotating frictional loads*, Proc. Inst. Mech. Engrs C, Vol. 208, (1994), pp. 417-425.

[40] L.G. Xiong, H. Chen, J.M. Yi, *Instability mechanism of a rotating disc subjected to various transverse interactive forces.* J. Mater. Proc. Techn., Vol. 129, No. 1, (2002), pp. 534-538.

[41] V.V. Kozlov, *Gyroscopic stabilization and parametric resonance*, J. Appl. Maths. Mechs., Vol. 65, No. 5, (2001), pp. 715-721.

[42] V.E. Shapiro, *Rotating class of parametric resonance processes in coupled oscillators*, Phys. Lett. A, Vol. 290, (2001), pp. 288-296.

[43] O.N. Kirillov, A.O. Seyranian, *The effect of small internal and external damping on the stability of distributed non-conservative systems*, J. Appl. Math. Mech., Vol. 69, No. 4, (2005), pp. 529-552.

[44] A.W. Leissa, *On a curve veering aberration*, Z. angew. Math. Phys., Vol. 25, (1974), pp. 99-111.

[45] S. Vidoli, F. Vestroni, *Veering phenomena in systems with gyroscopic coupling*, Trans. ASME, J. Appl. Mech., Vol. 72, (2005), pp. 641-647.

[46] V.M. Lakhadanov, *On stabilization of potential systems*, Prikl. Mat. Mekh., Vol. 39, No. 1, (1975), pp. 53-58.

[47] O.N. Kirillov, A.P. Seyranian, *Solution to the Herrmann-Smith problem*, Dokl. Phys., Vol. 47, No. 10, (2002), pp. 767-771.

[48] J. Kang, C.M. Krousgrill, F. Sadeghi, *Dynamic instability of a thin circular plate with friction interface and its application to disc brake squeal.* J. Sound. Vibr., (2008).

[49] J.D. Fieldhouse, N. Ashraf, C. Talbot, *The Measurement and Analysis of the Disc/Pad Interface Dynamic Centre of Pressure and its Influence on Brake Noise*, SAE Paper 2008-01-0826, in: SP-2188, Brake technology, 2008 (SAE, Warrendale, 2008).