Sensitivity analysis of gyroscopic and circulatory systems prone to dissipation-induced instabilities

Oleg N. Kirillov

Abstract Asymptotic stability is examined for a linear potential system perturbed by small gyroscopic, dissipative, and non-conservative forces as well as for a circulatory system with small velocity-dependent forces and for a gyroscopic system with small dissipative and circulatory forces. Typical singularities of the stability boundary are revealed that govern stabilization and destabilization and cause the imperfect merging of modes. Sensitivity analysis of the critical parameters is performed with the use of the perturbation theory for eigenvalues and eigenvectors of non-self-adjoint operators. In case of two degrees of freedom, stability boundary is found in terms of the invariants of matrices of the system. Bifurcation of the stability domain due to change of the structure of the damping matrix is described. As a mechanical example, the onset of stabilization and destabilization in the models of gyropendulums and of rotating continua in frictional contact is investigated

1 Introduction

We consider a non-conservative system depending on three parameters Ω, δ, and ν

$$\ddot{\mathbf{x}} + (\Omega\mathbf{G} + \delta\mathbf{D})\dot{\mathbf{x}} + (\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0 \tag{1}$$

with the real matrices $\mathbf{K} = \mathbf{K}^T$, $\mathbf{D} = \mathbf{D}^T$, $\mathbf{G} = -\mathbf{G}^T$, and $\mathbf{N} = -\mathbf{N}^T$ of potential, dissipative, gyroscopic, and non-conservative positional (circulatory) forces, where dot stands for the time differentiation and $\mathbf{x} \in \mathbb{R}^m$. A *circulatory* system

$$\ddot{\mathbf{x}} + (\mathbf{K} + \nu\mathbf{N})\mathbf{x} = 0 \tag{2}$$

as well as a *gyroscopic* one

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$$\ddot{\mathbf{x}} + \Omega \mathbf{G}\dot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0. \tag{3}$$

possess symmetries that are evident after transformation of (1) to $\dot{\mathbf{y}} = \mathbf{A}\mathbf{y}$ with

$$\mathbf{A}(\delta,\Omega,\nu) = \begin{bmatrix} -\frac{1}{2}\Omega\mathbf{G} & \mathbf{I} \\ \frac{1}{2}\delta\Omega\mathbf{D}\mathbf{G} + \frac{1}{4}\Omega^2\mathbf{G}^2 - \mathbf{K} - \nu\mathbf{N} & -\delta\mathbf{D} - \frac{1}{2}\Omega\mathbf{G} \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} \mathbf{x} \\ \dot{\mathbf{x}} + \frac{1}{2}\Omega\mathbf{G}\mathbf{x} \end{bmatrix},$$

where $\mathbf{I}$ is the identity. The matrix $\mathbf{A}(0,0,\nu)$ has a *reversible* symmetry $\mathbf{RAR} = -\mathbf{A}$ [8], while $\mathbf{A}(0,\Omega,0)$ possesses the *Hamiltonian* symmetry $\mathbf{JAJ} = \mathbf{A}^T$ [13], where

$$\mathbf{R} = \mathbf{R}^{-1} = \begin{bmatrix} \mathbf{I} & 0 \\ 0 & -\mathbf{I} \end{bmatrix}, \quad \mathbf{J} = -\mathbf{J}^{-1} = \begin{bmatrix} 0 & \mathbf{I} \\ -\mathbf{I} & 0 \end{bmatrix}. \tag{4}$$

In both cases $\det(\mathbf{A} - \lambda\mathbf{I}) = \det(\mathbf{A} + \lambda\mathbf{I})$, which implies marginal stability if all the eigenvalues are purely imaginary and semi-simple.

In the presence of all the four forces, the symmetries are broken and the marginal stability is generally destroyed. Instead, system (1) can be asymptotically stable. Modern applications in rotor dynamics, hydrodynamics, stability and optimization of structures, and acoustics of friction frequently lead to the linearized equations of motion (1) with $\delta,\Omega \ll \nu$ (*near-reversible system*), with $\delta,\nu \ll \Omega$ (*near-Hamiltonian system*), or with $\delta,\Omega,\nu \ll 1$ (*near-potential system*).

Historically, Thomson and Tait in 1879 were the first who found that dissipation destroys the gyroscopic stabilization (*dissipation-induced instability*) [13]. A similar effect of non-conservative positional forces on the stability of gyroscopic systems has been established by Lakhadanov [3]. A more sophisticated manifestation of the dissipation-induced instabilities has been discovered by Ziegler on the example of a double pendulum loaded by a follower force with the damping, non-uniformly distributed among the natural modes [1]. Without dissipation, the Ziegler pendulum is a reversible system, which is marginally stable for the loads non-exceeding some critical value. Small dissipation of order $o(1)$ makes the pendulum either unstable or asymptotically stable with the critical load, which generically is lower than that of the undamped system by the quantity of order $O(1)$ (*the destabilization paradox*). Similar discontinuous change in the stability domain for near-Hamiltonian systems has been observed by Holopainen [2, 13] in his study of the effect of dissipation on the stability of baroclinic waves in Earth's atmosphere, by Hoveijn and Ruijgrok on the example of a rotating shaft on an elastic foundation [6], and by Crandall, who investigated a gyroscopic pendulum with stationary and rotating damping [7].

As it was understood during the last decade, the reason for the destabilization paradox in multiparameter near-reversible and near-Hamiltonian systems is multiple eigenvalues related to the singularities of the stability boundary. Hoveijn and Ruijgrok were the first who associated the discontinuous change in the critical load in their example to the Whitney umbrella singularity [6]. The same singularity has been identified on the stability boundary of the Ziegler pendulum [11], of the models of disc brakes [16, 19], of the rods loaded by follower force [12], and of the gyropendulums and spinning tops [14, 18]. These examples reflect the general fact that the codimension-1 Hamiltonian (or reversible) Hopf bifurcation can be viewed as a singular limit of the codimension-3 dissipative resonant 1 : 1 normal form and the essential singularity in which these two cases meet is topologically equivalent to Whitney's umbrella [10, 13].

Despite the achieved qualitative understanding, the growing number of applications demonstrating the destabilization paradox [9, 16, 13, 17, 21] as well as the need for a justification for the use of Hamiltonian or reversible models to describe real-world systems that are in fact only near-Hamiltonian or near-reversible, requires the development of new analytical tools for a unified treatment of this phenomenon.

In the present paper we propose an efficient sensitivity analysis for calculation of the stability boundaries and for evaluation of critical parameters of system (1) in near-reversible, near-Hamiltonian, and near-potential cases.

2 A circulatory system with small velocity-dependent forces

We begin with the near-reversible case ($\delta, \Omega \ll \nu$).

Proposition 1. *If* $\mathrm{tr}\mathbf{K} > 0$ *and* $\det\mathbf{K} \le 0$, *circulatory system (2) with two degrees of freedom is stable for* ${\nu_d}^2 < \nu^2 < {\nu_f}^2$, *unstable by divergence for* $\nu^2 \le {\nu_d}^2$, *and unstable by flutter for* $\nu^2 \ge {\nu_f}^2$, *where the critical values* ν_d *and* ν_f *are*

$$0 \le \sqrt{-\det\mathbf{K}} =: \nu_d \le \nu_f := \sqrt{(\mathrm{tr}\mathbf{K}/2)^2 - \det\mathbf{K}}. \tag{5}$$

If $\mathrm{tr}\mathbf{K} > 0$ *and* $\det\mathbf{K} > 0$, *the circulatory system is stable for* $\nu^2 < {\nu_f}^2$ *and unstable by flutter for* $\nu^2 \ge {\nu_f}^2$. *If* $\mathrm{tr}\mathbf{K} \le 0$, *the system is unstable.*

At the flutter boundary $\nu = \nu_f$ there exist a double eigenvalue $i\omega_f$ with the right and left Jordan chains $\mathbf{u}_0$, $\mathbf{u}_1$ and $\mathbf{v}_0$, $\mathbf{v}_1$

$$\begin{aligned} (-\omega_f^2\mathbf{I} + \mathbf{K} + \nu_f\mathbf{N})\mathbf{u}_0 &= 0, \quad (-\omega_f^2\mathbf{I} + \mathbf{K} + \nu_f\mathbf{N})\mathbf{u}_1 = -2i\omega_f\mathbf{u}_0, \\ \mathbf{v}_0^T(-\omega_f^2\mathbf{I} + \mathbf{K} + \nu_f\mathbf{N}) &= 0, \quad \mathbf{v}_1^T(-\omega_f^2\mathbf{I} + \mathbf{K} + \nu_f\mathbf{N}) = -2i\omega_f\mathbf{v}_0^T, \end{aligned} \tag{6}$$

where $\omega_f = \sqrt{\mathrm{tr}\mathbf{K}/2}$ for $m = 2$. For $\nu > \nu_f$ the flutter instability is caused by two of the four complex eigenvalues lying on the branches of a hyperbolic curve $\mathrm{Im}\lambda^2 - \mathrm{Re}\lambda^2 = \omega_f^2$, see Fig. 1. In the vicinity of $\nu = \nu_f$ we have [11]

$$\begin{aligned} \lambda(\nu) = i\omega_f \pm \mu\sqrt{\nu - \nu_f} + \dots, \quad \mathbf{u}(\nu) &= \mathbf{u}_0 \pm \mu\mathbf{u}_1\sqrt{\nu - \nu_f} + \dots \\ \mathbf{v}(\nu) &= \mathbf{v}_0 \pm \mu\mathbf{v}_1\sqrt{\nu - \nu_f} + \dots, \end{aligned} \tag{7}$$

where $\mu^2 = -\mathbf{v}_0^T\mathbf{N}\mathbf{u}_0(2i\omega_f\mathbf{v}_0^T\mathbf{u}_1)^{-1}$ is real. For $m = 2$ we have $\mu^2 = \nu_f/2\omega_f^2 > 0$.

System (1) with $m = 2$ and $\det\mathbf{G} = \det\mathbf{N} = 1$ is asymptotically stable iff [14]

$$\delta\mathrm{tr}\mathbf{D} > 0,\ \mathrm{tr}\mathbf{K} + \delta^2\det\mathbf{D} + \Omega^2 > 0,\ \det\mathbf{K} + \nu^2 > 0,\ -(\nu - \nu_{cr}^-)(\nu - \nu_{cr}^+) > 0, \tag{8}$$

where $\nu_{cr}^{\pm}(\delta, \Omega) = a^{-1}(\Omega b \pm \sqrt{\Omega^2 b^2 + ac})\,\delta$, $\beta_* = (2\nu_f)^{-1}\mathrm{tr}(\mathbf{K}\mathbf{D} - \omega_f^2\mathbf{D})$, and

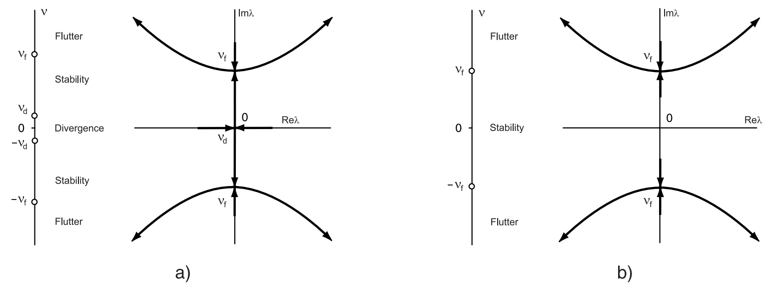

Fig. 1 Stability diagrams and trajectories of eigenvalues for the increasing parameter $\nu > 0$ for the circulatory system (2) with $\mathrm{tr}\mathbf{K} > 0$ and $\det\mathbf{K} < 0$ (a) and $\mathrm{tr}\mathbf{K} > 0$ and $\det\mathbf{K} > 0$ (b).

$$
\begin{aligned}
a(\delta,\Omega) &= 4\Omega^2 + \delta^2(\mathrm{tr}\mathbf{D})^2, \quad b(\delta,\Omega) = 4\nu_f\beta_* + (\delta^2\det\mathbf{D} + \Omega^2)\mathrm{tr}\mathbf{D},\\
c(\delta,\Omega) &= \nu_f^2((\mathrm{tr}\mathbf{D})^2 - 4\beta_*^2) + (\omega_f^2\mathrm{tr}\mathbf{D} - 2\nu_f\beta_*)(\delta^2\det\mathbf{D} + \Omega^2)\mathrm{tr}\mathbf{D}.
\end{aligned} \tag{9}
$$

Calculating $\nu_0^{\pm}(\beta) = \lim_{\delta\to 0}\nu_{cr}^{\pm}(\delta,\beta\delta)$ and then isolating β yields a linear approximation to the stability boundary in the vicinity of the ν-axis

$$
\Omega = \frac{\nu_f}{\nu}\left[\beta_* \pm \frac{\mathrm{tr}\mathbf{D}}{2}\sqrt{1 - \frac{\nu^2}{\nu_f^2}}\right]\delta. \tag{10}
$$

According to (10) the configuration of the domain of asymptotic stability depends on the structure of the matrix $\mathbf{D}$. Due to the identity

$$
\beta_*^2 - \frac{(\mathrm{tr}\mathbf{D})^2}{4} = -\det\mathbf{D} - \frac{(k_{12}(d_{22} - d_{11}) - d_{12}(k_{22} - k_{11}))^2}{4\nu_f^2} \tag{11}
$$

the set of indefinite damping matrices is subdivided into two classes.

Definition 1. We call a 2×2 real symmetric matrix $\mathbf{D}$ with $\det\mathbf{D} < 0$ *weakly indefinite*, if $4\beta_*^2 < (\mathrm{tr}\mathbf{D})^2$, and *strongly indefinite*, if $4\beta_*^2 > (\mathrm{tr}\mathbf{D})^2$.

For $\mathbf{K} > 0$ and a positive (semi)definite or a weakly-indefinite matrix $\mathbf{D}$ the addition of small velocity-dependent forces blows the stability interval of a circulatory system $\nu^2 < \nu_f^2$ up to a three-dimensional region bounded by the parts of a singular surface $\nu = \nu_{cr}^{\pm}(\delta,\Omega)$, which belong to the half-space $\delta\mathrm{tr}\mathbf{D} > 0$, Fig. 2(a). The interval $\nu^2 < \nu_f^2$ forms an edge of a dihedral angle. At $\nu = 0$ the angle reaches its maximum (π), creating another edge along the Ω-axis. While approaching the points $\pm\nu_f$, the angle becomes more acute. Leaving only the second order terms and then substituting $\beta = \Omega/\delta$ in the expansions following from (10)

$$
\nu_f \mp \nu_0^{\pm}(\beta) = 2\nu_f(\mathrm{tr}\mathbf{D})^{-2}(\beta \mp \beta_*)^2 + o((\beta \mp \beta_*)^2), \tag{12}
$$

we get equations of the form $Z = X^2/Y^2$, which is canonical for the Whitney umbrella surface [6, 10]. The extension to arbitrary m is provided by the statement.

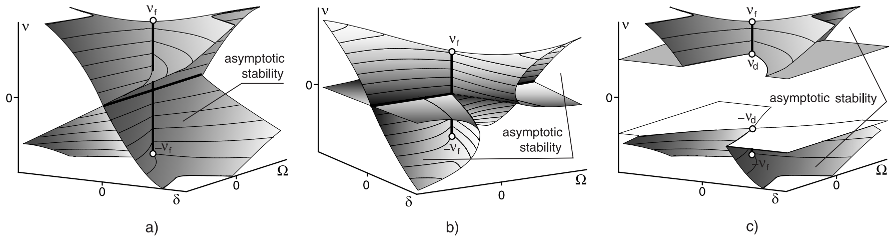

Fig. 2 The domain of asymptotic stability of the two-dimensional system (1) with the singularities *Whitney umbrella*, *dihedral angle*, *trihedral angle*, and *break of an edge* when $\mathbf{K} > 0$ and $4\beta_*^2 < (\mathrm{tr}\mathbf{D})^2$ (a), $\mathbf{K} > 0$ and $4\beta_*^2 > (\mathrm{tr}\mathbf{D})^2$ (b), and when $\mathrm{tr}\mathbf{K} > 0$ and $\det\mathbf{K} < 0$ (c).

Theorem 1. *Let the system (2) with m degrees of freedom be stable for* $\nu < \nu_f$. *Define the real quantities* $\beta_* = -\mathbf{v}_0^T\mathbf{D}\mathbf{u}_0(\mathbf{v}_0^T\mathbf{G}\mathbf{u}_0)^{-1}$ *and*

$$\begin{aligned} d_1 &= \mathrm{Re}(\mathbf{v}_0^T\mathbf{D}\mathbf{u}_0), \quad d_2 = \mathrm{Im}(\mathbf{v}_0^T\mathbf{D}\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{D}\mathbf{u}_0), \\ g_1 &= \mathrm{Re}(\mathbf{v}_0^T\mathbf{G}\mathbf{u}_0), \quad g_2 = \mathrm{Im}(\mathbf{v}_0^T\mathbf{G}\mathbf{u}_1 + \mathbf{v}_1^T\mathbf{G}\mathbf{u}_0). \end{aligned} \tag{13}$$

Then, in the vicinity of $\beta := \Omega/\delta = \beta_*$ *the limit of the critical flutter load* ν_{cr}^+ *of the near-reversible system (1) with m degrees of freedom as* $\delta \to 0$ *is*

$$\nu_0^+(\beta) = \nu_f - \frac{g_1^2(\beta-\beta_*)^2}{\mu^2(d_2+\beta_* g_2)^2} + o((\beta-\beta_*)^2) \le \nu_f. \tag{14}$$

Proof. Perturbing a simple eigenvalue $i\omega(\nu)$ of the stable system (2) at a fixed $\nu < \nu_f$ by small dissipative and gyroscopic forces yields the increment

$$\lambda = i\omega - \frac{\mathbf{v}^T\mathbf{D}\mathbf{u}}{2\mathbf{v}^T\mathbf{u}}\delta - \frac{\mathbf{v}^T\mathbf{G}\mathbf{u}}{2\mathbf{v}^T\mathbf{u}}\Omega + o(\delta,\Omega). \tag{15}$$

Substituting expansions (7) into (15) and equating the real first order increment to zero yields expression (14) for $|\beta - \beta_*| \ll 1$, which for $m = 2$ is reduced to (12). □

When $\mathbf{D}$ approaches the threshold $4\beta_*^2 = (\mathrm{tr}\mathbf{D})^2$, two smooth parts of the stability boundary come towards each other until they touch, when $\mathbf{D}$ is at the threshold. After $\mathbf{D}$ becomes strongly indefinite this temporary configuration collapses into two pockets of asymptotic stability, as shown in Fig. 2(b). Each of the two pockets has a Whitney umbrella as well as two edges which meet at the origin and form the "break of an edge" singularity. In case of an indefinite matrix $\mathbf{K}$ the condition $\nu^2 > \nu_d^2$ divides the domain of asymptotic stability into two parts, Fig. 2(c). Qualitatively, this configuration does not depend on the properties of the matrix $\mathbf{D}$.

Note that the parameter $4\beta_*^2 - (\mathrm{tr}\mathbf{D})^2$ obtained from the local perturbation analysis governs the global bifurcation of the whole stability domain, which is seen at $\nu = 0$ when the stability domain is described by the inequality $c(\delta,\Omega) > 0$, Fig. 3.

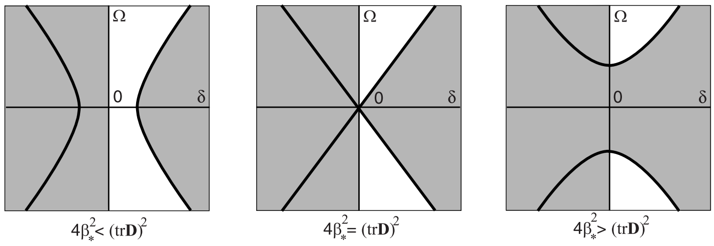

Fig. 3 Bifurcation of the domain of the asymptotic stability (white) in the plane (δ, Ω) at $\nu = 0$ with the change of the matrix $\mathbf{D}$ from the positive definite or weakly indefinite to strongly indefinite.

Inequalities (8) describe the stability domain of $m = 2$-dimensional near-potential system, shown in Fig. 2(a,b). In case of arbitrary m approximation to the domain is captured by the first- and second-order terms in the Taylor series for simple eigenvalues [5] of the matrix $\mathbf{K}$ perturbed by the gyroscopic, dissipative and circulatory forces. The case when $\mathbf{K}$ has repeated eigenvalues will be considered in Section 4.

3 A gyroscopic system with weak damping and circulatory forces

Stability of a two-dimensional gyroscopic system (3) in the absence of dissipative and circulatory forces ($\delta = \nu = 0$) is given by the following statement.

Proposition 2. *If* $\det\mathbf{K} > 0$ *and* $\mathrm{tr}\mathbf{K} < 0$*, gyroscopic system (3) with two degrees of freedom is unstable by divergence for* $\Omega^2 < {\Omega_0^-}^2$*, unstable by flutter for* ${\Omega_0^-}^2 \leq \Omega^2 \leq {\Omega_0^+}^2$*, and stable for* ${\Omega_0^+}^2 < \Omega^2$*, where the critical values* Ω_0^- *and* Ω_0^+ *are*

$$0 \leq \sqrt{-\mathrm{tr}\mathbf{K} - 2\sqrt{\det\mathbf{K}}} =: \Omega_0^- \leq \Omega_0^+ := \sqrt{-\mathrm{tr}\mathbf{K} + 2\sqrt{\det\mathbf{K}}}. \tag{16}$$

If $\det\mathbf{K} > 0$ *and* $\mathrm{tr}\mathbf{K} > 0$*, the gyroscopic system is stable for any* Ω*.*
If $\det\mathbf{K} \leq 0$*, the system is unstable.*

For $\mathbf{K} < 0$ the statically unstable potential system can be stabilized by the gyroscopic forces. With the increase of Ω^2 the complex eigenvalues move along the circle $\mathrm{Re}\lambda^2 + \mathrm{Im}\lambda^2 = \omega_0^2 = \sqrt{\det\mathbf{K}}$ until at $\Omega^2 = {\Omega_0^+}^2$ they reach the imaginary axis and originate double eigenvalues $\pm i\omega_0$, Fig. 4. The onset of the gyroscopic stabilization $\Omega_{cr}(\delta, \nu)$ of the near-Hamiltonian system deviates from Ω_0^+ [18].

Theorem 2. *Let the system (3) with even number m of degrees of freedom be gyroscopically stabilized for* $\Omega > \Omega_0^+$ *and let at* $\Omega = \Omega_0^+$ *its spectrum contain a double eigenvalue* $i\omega_0$ *with the Jordan chain* $\mathbf{u}_0$*,* $\mathbf{u}_1$*, satisfying the equations*

$$\begin{aligned} (-\mathbf{I}\omega_0^2 + i\omega_0\Omega_0^+\mathbf{G} + \mathbf{K})\mathbf{u}_0 &= 0, \\ (-\mathbf{I}\omega_0^2 + i\omega_0\Omega_0^+\mathbf{G} + \mathbf{K})\mathbf{u}_1 &= -(2i\omega_0\mathbf{I} + \Omega_0^+\mathbf{G})\mathbf{u}_0. \end{aligned} \tag{17}$$

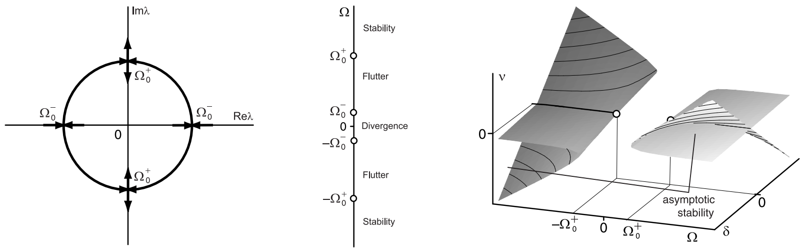

Fig. 4 The gyroscopic system with $\mathbf{K} < 0$: trajectories of the eigenvalues for the increasing parameter $\Omega > 0$ (left), stability diagram (middle), the domain of asymptotic stability of the near-hamiltonian system with the Whitney umbrella singularities (right).

Define the real quantities $\gamma_* = -i\omega_0 \overline{\mathbf{u}}_0^T \mathbf{D}\mathbf{u}_0 (\overline{\mathbf{u}}_0^T \mathbf{N}\mathbf{u}_0)^{-1}$ *and* d_1, d_2, n_1, n_2 *as*

$$\begin{aligned} d_1 &= \mathrm{Re}(\overline{\mathbf{u}}_0^T \mathbf{D}\mathbf{u}_0), \quad d_2 = \mathrm{Im}(\overline{\mathbf{u}}_0^T \mathbf{D}\mathbf{u}_1 - \overline{\mathbf{u}}_1^T \mathbf{D}\mathbf{u}_0), \\ n_1 &= \mathrm{Im}(\overline{\mathbf{u}}_0^T \mathbf{N}\mathbf{u}_0), \quad n_2 = \mathrm{Re}(\overline{\mathbf{u}}_0^T \mathbf{N}\mathbf{u}_1 - \overline{\mathbf{u}}_1^T \mathbf{N}\mathbf{u}_0), \end{aligned} \tag{18}$$

where the bar over a symbol denotes complex conjugate. Then, in the vicinity of $\gamma := \nu/\delta = \gamma_*$ *the limit of the critical value of the gyroscopic parameter* Ω_{cr}^+ *of the near-Hamiltonian system as* $\delta \to 0$ *is*

$$\Omega_{cr}^+(\gamma) = \Omega_0^+ + \frac{n_1^2(\gamma-\gamma_*)^2}{\mu^2(\omega_0 d_2 - \gamma_* n_2 - d_1)^2} + o((\gamma-\gamma_*)^2) \geq \Omega_0^+. \tag{19}$$

With $\gamma = \nu/\delta$ expression (19) gives an approximation to the stability boundary of the near-Hamiltonian system in the vicinity of the Whitney umbrella singularity, Fig. 4. In case of $m = 2$ degrees of freedom the estimate (19) is reduced to

$$\pm\Omega_{cr}^+(\gamma) = \pm\Omega_0^+ \pm \Omega_0^+ \frac{2}{(\omega_0 \mathrm{tr}\mathbf{D})^2}(\gamma \mp \gamma_*)^2 + o((\gamma \mp \gamma_*)^2) \tag{20}$$

with $\gamma_* = (2\Omega_0^+)^{-1}\mathrm{tr}[\mathbf{K}\mathbf{D} + ({\Omega_0^+}^2 - \omega_0^2)\mathbf{D}]$, see e.g. [14, 18]. For $\mathbf{K} > 0$ the domain of asimptotic stability has a shape of the twisted dihedral angle, Fig. 2(a,b).

An instructive illustration to the general analysis is the domain of asymptotic stability of the modified Maxwell-Bloch equations that are the normal form for rotationally symmetric, planar dynamical systems [13] and describe, in particular, stability of the vertical equilibrium of the Hauger gyropendulum [4, 20]. The equations follow from (1) for $m = 2$, $\mathbf{D} = \mathbf{I}$, and $\mathbf{K} = \kappa\mathbf{I}$, and can be written as

$$\ddot{x} + i\Omega\dot{x} + \delta\dot{x} + i\nu x + \kappa x = 0, \quad x = x_1 - ix_2. \tag{21}$$

According to (8) the solution $x = 0$ of (21) is asymptotically stable if and only if $\delta > 0$ and $\Omega > \nu/\delta - (\delta/\nu)\kappa$. For $\kappa > 0$ the stability domain is a twisted dihedral angle causing gyroscopic destabilization, Fig. 5(a). For $\kappa < 0$ the domain of asymptotic stability collapses into two disjoint parts that are pockets of two Whitney umbrellas

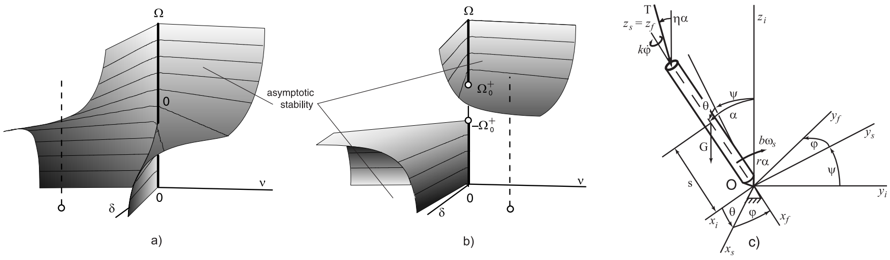

Fig. 5 Two configurations of the asymptotic stability domain of the modified Maxwell-Bloch equations for $\kappa > 0$ (a) and $\kappa < 0$ (b) corresponding to *gyroscopic destabilization* and *gyroscopic stabilization* respectively; Hauger's gyropendulum (c).

singled out by inequality $\delta > 0$. Consequenctly, the system unstable at $\Omega = 0$ can become asymptotically stable at greater Ω, as shown in Fig. 5(b) by the dashed line.

4 Subcritical flutter of rotating continua in fricitonal contact

Equations (1) with $\Omega = 2\widetilde{\Omega}$ and $\mathbf{K} = (\rho^2 - \widetilde{\Omega}^2)\mathbf{I}$ describe a two-mode approximation of the models of rotating continua in frictional contact [15, 19]. In the absence of dissipative and non-conservative positional forces the eigenvalues $\lambda_p^\pm = i\rho \pm i\widetilde{\Omega}$, $\lambda_n^\pm = -i\rho \pm i\widetilde{\Omega}$ of the operator $\mathbf{L}_0(\widetilde{\Omega}) = \mathbf{I}\lambda^2 + 2\lambda\widetilde{\Omega}\mathbf{G} + (\rho^2 - \widetilde{\Omega}^2)\mathbf{I}$ form *a spectral mesh* [19] in the plane $(\widetilde{\Omega}, \mathrm{Im}\lambda)$. Two nodes of the mesh at $\widetilde{\Omega} = \widetilde{\Omega}_0 = 0$ correspond to the double semi-simple eigenvalues $\lambda = \pm i\rho$.

Consider a perturbation $\mathbf{L}_0(\widetilde{\Omega}) + \Delta\mathbf{L}(\widetilde{\Omega})$, assuming that $\Delta\mathbf{L}(\widetilde{\Omega}) = \delta\lambda\mathbf{D} + \nu\mathbf{N} \sim \varepsilon$. For small $\widetilde{\Omega}$ and $\varepsilon = \|\Delta\mathbf{L}(0)\|$ the perturbed eigenvalue $i\rho$ is [20]

$$\mathrm{Re}\lambda = -\frac{\mu_1 + \mu_2}{4}\delta \pm \sqrt{\frac{|c| + \mathrm{Re}c}{2}}, \quad \mathrm{Im}\lambda = \rho \pm \sqrt{\frac{|c| - \mathrm{Re}c}{2}}, \tag{22}$$

where μ_1 and μ_2 are the eigenvalues of $\mathbf{D}$, and

$$\mathrm{Re}c = \left(\frac{\mu_1 - \mu_2}{4}\right)^2 \delta^2 - \widetilde{\Omega}^2 + \frac{\nu^2}{4\rho^2}, \quad \mathrm{Im}c = \frac{\widetilde{\Omega}\nu}{\rho}. \tag{23}$$

According to (22) independently on the structure of the matrix $\mathbf{D}$, the primary role of dissipation is the creation of the bubble of instability. It is submerged below the surface $\mathrm{Re}\lambda = 0$ in the space $(\widetilde{\Omega}, \mathrm{Im}\lambda, \mathrm{Re}\lambda)$ in case of full dissipation Fig. 6(a) and partially lies in the domain $\mathrm{Re}\lambda > 0$ when damping is indefinite Fig. 6(b). In contrast to the effect of indefinite damping, the non-conservative positional forces destabilize one half of the modes simultaneously at every Ω, Fig. 6(d). In order to localize the instability, a combination of circulatory and dissipative forces is re-

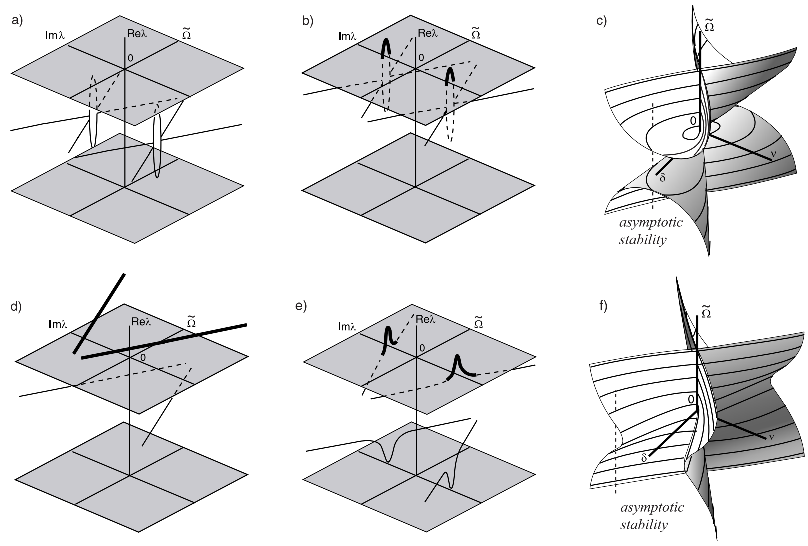

Fig. 6 The mechanism of subcritical flutter: The ring (bubble) of complex eigenvalues under the surface $\mathrm{Re}\lambda = 0$ due to action of full dissipation with $\det \mathbf{D} > 0$ - a latent source of instability (a); emersion of the bubble of instability due to indefinite damping with $\det \mathbf{D} < 0$ (b); repulsion of eigenvalue branches of the spectral mesh due to action of circulatory forces (d); collapse of the bubble of instability and immersion and emersion of its parts due to action of dissipative and circulatory forces (e); stability domains in case of indefinite damping (c) and full dissipation (f).

quired. Non-conservative positional forces destroy the bubble into two branches and shift one of them to the region of positive real parts even in case of full dissipation Fig. 6(e). Since the branch remembers the existence of the bubble, the subcritical flutter, which is responsible for the onset of squeal in brakes, is developing near the nodes of the spectral mesh at a frequency $\omega_{cr}^- < \omega < \omega_{cr}^+$ when $\widetilde{\Omega}^2 < \widetilde{\Omega}_{cr}^2$ with

$$\widetilde{\Omega}_{cr} = \delta \frac{\mathrm{tr}\mathbf{D}}{4} \sqrt{\frac{\delta^2 \rho^2 \det \mathbf{D} - \nu^2}{\nu^2 - \delta^2 \rho^2 (\mathrm{tr}\mathbf{D}/2)^2}}, \quad \omega_{cr}^{\pm} = \rho \pm \frac{\nu}{2\rho} \sqrt{\frac{\delta^2 \rho^2 \det \mathbf{D} - \nu^2}{\nu^2 - \delta^2 \rho^2 (\mathrm{tr}\mathbf{D}/2)^2}}. \quad (24)$$

The first of equations (24) approximates the boundary of the domain of asymptotic stability, which bifurcates with the change of sign of $\det \mathbf{D}$, Fig. 6(c,e).

Conclusions

Investigation of stability and sensitivity analysis of the critical parameters of near-Hamiltonian, near-reversible, and near-potential systems is complicated by the singularities of the boundary of the domain of asymptotic stability, which can bifurcate due to change of the structure of the matrices involved. The proposed approach is an efficient analytical tool for the description of the mechanisms of stabilization and destabilization in the modern problems of rotor dynamics and acoustics of friction of rotating continua, containing circulatory forces and indefinite damping.

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References

1. Ziegler, H.: Die Stabilitätskriterien der Elastomechanik. Ing.-Arch. **20**, 49–56 (1952).
2. Holopainen, E.O.: On the effect of friction in baroclinic waves. Tellus. **13**(3), 363–367 (1961).
3. Lakhadanov, V.M.: On stabilization of potential systems. PMM **39**(1), 53–58 (1975).
4. Hauger, W.: Stability of a gyroscopic non-conservative system. Trans. ASME, J. Appl. Mech. **42**, 739–740 (1975).
5. Stewart, G.W., Sun, J.G.: Matrix Perturbation Theory. Academic Press, Boston (1990).
6. Hoveijn, I., Ruijgrok, M.: The stability of parametrically forced coupled oscillators in sum resonance. Z. angew. Math. Phys. **46**, 384–392 (1995).
7. Crandall, S.H.: The effect of damping on the stability of gyroscopic pendulums. Z. angew. Math. Phys. **46**, 761–780 (1995).
8. O'Reilly, O.M., Malhotra, N.K., Namachchivaya, N.S.: Some aspects of destabilization in reversible dynamical systems with application to follower forces. Nonlin. Dyn. **10**, 63–87 (1996).
9. Hoffmann, N., Gaul, L.: Effects of damping on mode-coupling instability in friction induced oscillations. Z. angew. Math. Mech. **83**, 524–534 (2003).
10. Langford, W.F.: Hopf meets Hamilton under Whitney's umbrella. in IUTAM symposium on nonlinear stochastic dynamics. Monticello, IL, USA, Augsut 26-30, 2002, Solid Mech. Appl. 110, S. N. Namachchivaya, et al., eds., Kluwer, Dordrecht, 157–165 (2003).
11. Kirillov, O.N.: A theory of the destabilization paradox in non-conservative systems. Acta Mechanica. **174**(3–4), 145–166 (2005).
12. Kirillov, O.N., Seyranian, A.P.: Instability of distributed nonconservative systems caused by weak dissipation. Dokl. Math. **71**(3), 470–475 (2005).
13. Krechetnikov, R., Marsden, J.E.: Dissipation-induced instabilities in finite dimensions. Rev. Mod. Phys. **79**, 519–553 (2007).
14. Kirillov, O.N.: Destabilization paradox due to breaking the Hamiltonian and reversible symmetry. Int. J. of Non-Lin. Mech., **42**(1), 71–87 (2007).
15. Spelsberg-Korspeter, G., Hochlenert, D., Kirillov, O.N., Hagedorn, P.: In- and out-of-plane vibrations of a rotating plate with frictional contact: Investigations on squeal phenomena. Trans. ASME, J. Appl. Mech., (2008), in press.
16. Sinou, J.-J., Jezequel, L.: Mode coupling instability in friction-induced vibrations and its dependency on system parameters including damping. Eur. J. Mech. A., **26**, 106–122 (2007).
17. Kessler, P., O'Reilly, O.M., Raphael, A.-L., Zworski, M.: On dissipation-induced destabilization and brake squeal: A perspective using structured pseudospectra. J. Sound Vibr., **308**, 1-11 (2007).
18. Kirillov, O.N.: Gyroscopic stabilization in the presence of non-conservative forces. Dokl. Math. **76**(2), 780–785 (2007).
19. Kirillov, O.N.: Bifurcation of the roots of the characteristic polynomial and destabilization paradox in friction induced oscillations. Theor. Appl. Mech. **34**(2), 87–109 (2007).
20. Kirillov, O.N.: How to Play a Disc Brake: A Dissipation-Induced Squeal. SAE Paper 2008-01-1160, SAE World Congress and Exhibition, April 2008, Detroit, MI, USA.
21. Spelsberg-Korspeter, G., Kirillov, O.N., Hagedorn, P.: Modeling and stability analysis of an axially moving beam with frictional contact: Trans. ASME, J. Appl. Mech. **75**(3), 031001 (2008).