How do small velocity-dependent forces (de)stabilize a non-conservative system?

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Abstract

The influence of small velocity-dependent forces on the stability of a linear autonomous non-conservative system of general type is studied. The problem is investigated by an approach based on the analysis of multiple roots of the characteristic polynomial whose coefficients are expressed through the invariants of the matrices of a non-conservative system. For systems with two degrees of freedom approximations of the domain of asymptotic stability are constructed and the structure of the matrix of velocity-dependent forces stabilizing a circulatory system is found. As mechanical examples the Bolotin problem and the Herrman–Jong pendulum are considered in detail.

1 Stabilization of a circulatory system

1.1 Introduction

We consider a linear autonomous non-conservative system

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{D}\frac{d\mathbf{y}}{dt} + \mathbf{A}\mathbf{y} = 0, \qquad (1)$$

where $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{A}$ are real 2×2 matrices of mass, damping and gyroscopic forces as well as non-conservative positional (circulatory) forces. Assuming that $\mathbf{y} = \mathbf{u}e^{\lambda t}$ we arrive at the eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{A})\mathbf{u} = 0. \qquad (2)$$

Stability of system (1) depends on the loci of the roots of the characteristic polynomial $P(\lambda) = \det(\lambda^2\mathbf{M}+\lambda\mathbf{D}+\mathbf{A})$ of eigenvalue problem (2) in the complex plane. In the absence of the velocity-dependent forces the *circulatory* [1] system

$$\mathbf{M}\frac{d^2\mathbf{y}}{dt^2} + \mathbf{A}\mathbf{y} = 0 \qquad (3)$$

can never be asymptotically stable, but it can be marginally stable oscillating with the limited amplitude. An introduction of the velocity dependent forces with the matrix $\mathbf{D}$ may lead the circulatory system both to the asymptotic stability and to the instability. That depends on a structure of the matrix $\mathbf{D}$. The goal of our paper is to express the conditions of asymptotic stability of the linear circulatory system perturbed by the velocity-dependent forces $\mathbf{D}$, directly in terms of the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{A}$.

1.2 Asymptotic stability domain

As it follows from the Appendix the characteristic polynomial of eigenvalue problem (2) has a compact form convenient for investigation of stability

$$P(\lambda) = \det\mathbf{M}\lambda^4 + \mathrm{tr}(\mathbf{D}^\dagger\mathbf{M})\lambda^3 + (\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}) + \det\mathbf{D})\lambda^2 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{D})\lambda + \det\mathbf{A}, \qquad (4)$$

where $\mathbf{D}^\dagger$, $\mathbf{A}^\dagger$ are the matrices adjoint to $\mathbf{D}$, $\mathbf{A}$

$$\mathbf{D}^\dagger = \begin{bmatrix} d_{22} & -d_{12} \\ -d_{21} & d_{11} \end{bmatrix}, \quad \mathbf{A}^\dagger = \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$

In the case of $m > 2$ degrees of freedom the main tool allowing to represent a characteristic polynomial by means of the invariants of the matrices involved is the Leverrier–Faddejev algorithm, see Appendix.

Assume that for $\mathbf{D} = 0$ the spectrum of the eigenvalue problem

$$(\mathbf{M}\lambda^2 + \mathbf{A})\mathbf{u} = 0 \qquad (5)$$

corresponding to the unperturbed circulatory system (3) consists of the double purely imaginary eigenvalues $\lambda_0 = \pm i\omega_0$ ($\omega_0 \neq \infty$) with the Jordan chain of length 2. The necessary and sufficient conditions for the roots $\pm i\omega_0$ ($\omega_0 \neq \infty$) of the polynomial

$$P(\lambda) = \det\mathbf{M}\lambda^4 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{M})\lambda^2 + \det\mathbf{A}, \qquad (6)$$

to be double are $\det\mathbf{M} \neq 0$ and $(\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}))^2 = 4\det\mathbf{M}\det\mathbf{A}$. As a consequence we have

$$\det\mathbf{A} = \omega_0^4\det\mathbf{M}, \quad \mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}) = 2\omega_0^2\det\mathbf{M}. \qquad (7)$$

Circulatory system (3), (7) belongs to the boundary between the stable systems and dynamically unstable (flutter) systems [1, 2] and is therefore unstable.

How should one choose the matrix $\mathbf{D}$ of the velocity-dependent forces to make circulatory system (3), (7) asymptotically stable?

Applying the Routh-Hurwitz criterion of asymptotic stability to polynomial (4) and taking into account conditions (7) we find the inequalities describing the asymptotic stability domain in the vicinity of the circulatory system (3), (7)

$$\det \mathbf{M} > 0, \quad \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}>0, \quad \mathrm{tr}\mathbf{A}^\dagger\mathbf{D}>0,$$

$$2\omega_0^2 \det \mathbf{M} + \det \mathbf{D}>0, \quad \omega_0^4 \det \mathbf{M} > 0, \qquad (8)$$

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}\mathrm{tr}\mathbf{D}^\dagger\mathbf{M} \det \mathbf{D}> \det \mathbf{M}(\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2.$$

Further we will assume that the matrix $\mathbf{M}$ has $\det \mathbf{M}>0$. In this case inequalities (8) are equivalent to the following three conditions

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}\ \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}\ \det \mathbf{D} > \det \mathbf{M}\ (\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2.$$

$$\det \mathbf{D} > 0, \quad \mathrm{tr}\mathbf{D}^\dagger\mathbf{M} > 0. \qquad (9)$$

Lemma 1 *Inequalities (9) are the necessary and sufficient conditions for the matrix* $\mathbf{D}$ *of the velocity-dependent forces to make circulatory system (3) with the matrices* $\mathbf{A}$ *and* $\mathbf{M}$ $(\det \mathbf{M}>0)$ *satisfying conditions (7) asymptotically stable.*

One can see from inequalities (9) that three parameters $\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}$, $\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}$, and $\det \mathbf{D}$ naturally appear in the stability conditions. The asymptotic stability boundary is therefore a surface

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}\ \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}\ \det \mathbf{D}= \det \mathbf{M}\ (\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2,$$

$$\det \mathbf{D}>0, \quad \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}>0 \qquad (10)$$

in the space of the parameters $\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}$, $\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}$, and $\det \mathbf{D}$, Figure 1.

Let us show that for small velocity-dependent forces this surface is the well-known Whitney umbrella [3]. For this purpose we assume $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$, where $\epsilon\geq 0$ is a small parameter. From Eq.(10) we can find that on the boundary of the asymptotic stability domain

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D} = \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}\left(\omega_0^2 - \frac{\epsilon^2}{2}\frac{\det\widetilde{\mathbf{D}}}{\det\mathbf{M}} \pm\right.$$

$$\left.\pm\, \epsilon\omega_0\sqrt{\frac{\det\widetilde{\mathbf{D}}}{\det\mathbf{M}}}\sqrt{1+\epsilon^2\frac{\det\widetilde{\mathbf{D}}}{4\omega_0^2\det\mathbf{M}}}\right) =$$

$$= \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}(\omega_0^2 \pm \epsilon\omega_0\sqrt{\det\widetilde{\mathbf{D}}/\det\mathbf{M}+O(\epsilon^2)}). \qquad (11)$$

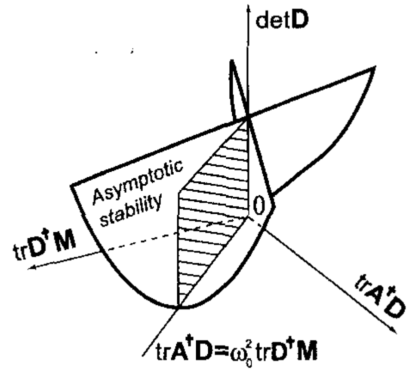

Figure 1: The geometrical meaning of the Routh-Hurwitz conditions (9): The Whitney umbrella (the tangent cone is hatched).

Eq.(11) without term $O(\epsilon^2)$ rewritten in the form

$$(\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2 = \frac{\det\mathbf{D}}{\det\mathbf{M}}\omega_0^2(\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2 \qquad (12)$$

describes the surface $XY^2{=}Z^2$ known as the Whitney umbrella [3] with $X{=}\det\mathbf{D}/\det\mathbf{M}$, $Y{=}\omega_0\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}$, $Z{=}\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}{-}\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}$.

Thus, for fairly small perturbations $\mathbf{D}$ asymptotic stability domain (9) in the neighborhood of the circulatory system (3), (7) is approximated by the following inequalities

$$\mathrm{tr}\mathbf{D}^\dagger\mathbf{M} > 0, \quad \det\mathbf{D} > 0,$$

$$\frac{\det\mathbf{D}}{\det\mathbf{M}}\omega_0^2(\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2 > (\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M})^2. \qquad (13)$$

One can see that the asymptotic stability exists inside of a half of the Whitney umbrella, shown in Figure 1 in the space of the parameters $\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}$, $\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}$, and $\det\mathbf{D}$. At the origin the stability domain has a generic singularity "deadlock of an edge" [3], corresponding to the double eigenvalue $i\omega_0$ of the matrix pencil $\mathbf{M}\lambda^2+\mathbf{A}$ of the unperturbed circulatory system (3).

1.3 Tangent cone to the stability domain and the structure of a stabilizing perturbation

Consider the plane $\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}=0$ in the space of parameters $\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}$, $\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}$, and $\det\mathbf{D}$. The part of this plane defined by the inequalities $\det\mathbf{D}>0$, $\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}>0$ belongs to the asymptotic stability domain (9). Indeed, in this case the first of inequalities (9) is satisfied because $\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}=\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M} > 0$. We thus have proved the following

Lemma 2 *Sufficient conditions for a* 2×2 *matrix* $\mathbf{D}$ *to stabilize a circulatory system (3) with the given*

2×2 *matrices* $\mathbf{A}$ *and* $\mathbf{M}$ ($\det\mathbf{M}>0$) *satisfying Eq.(7) are*

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}-\omega_0^2\mathrm{tr}\mathbf{D}^\dagger\mathbf{M}=0, \qquad (14)$$

$$\det\mathbf{D}>0,\quad \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}>0. \qquad (15)$$

Consider again the asymptotic stability domain (9) in a small neighborhood of the origin. Assuming $\mathbf{D}=\epsilon\widetilde{\mathbf{D}}$ where $\epsilon\geq0$ is a small parameter we rewrite the first of inequalities (9) in the form

$$\mathrm{tr}\mathbf{A}^\dagger\widetilde{\mathbf{D}}\mathrm{tr}\widetilde{\mathbf{D}}^\dagger\mathbf{M}\det\widetilde{\mathbf{D}}\epsilon^4>\epsilon^2\det\mathbf{M}\,(\mathrm{tr}\mathbf{A}^\dagger\widetilde{\mathbf{D}}-\omega_0^2\mathrm{tr}\widetilde{\mathbf{D}}^\dagger\mathbf{M})^2.$$

One can see that for $\epsilon\to0$ this condition is satisfied only for the matrices $\widetilde{\mathbf{D}}$ lying in the plane $\mathrm{tr}\mathbf{A}^\dagger\widetilde{\mathbf{D}}-\omega_0^2\mathrm{tr}\widetilde{\mathbf{D}}^\dagger\mathbf{M}=0$. This means that in the vicinity of the origin the asymptotic stability domain (9) is very narrow, so it is well approximated by the set given by Eqs.(14), (15), which in fact is *a tangent cone* to this domain, i.e. a set of vectors starting at the origin and lying in the domain. Such a geometry of the asymptotic stability domain is responsible for the destabilization paradox due to small damping [1, 4] because in the generic case a small perturbation $\mathbf{D}$ leads to instability. The sufficient conditions given by Lemma 2 allow us to find the stabilizing velocity-dependent perturbation in an explicit form.

Lemma 3 *Velocity-dependent forces with the matrix*

$$\mathbf{D}=\sum_{p=-\infty}^{\infty}c_p\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p,\quad \det\mathbf{M}>0,\quad c_p\geq0, \qquad (16)$$

where p is an integer index, make circulatory system (3),(7) asymptotically stable.

Proof: For the proof it's enough to consider $\mathbf{D}=\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p$. Using the identities

$$\mathbf{M}^\dagger\mathbf{M}=\mathbf{I}\det\mathbf{M},\quad \mathbf{A}^\dagger\mathbf{A}=\omega_0^4\mathbf{I}\det\mathbf{M},$$

where $\mathbf{I}$ is the identity matrix we rewrite $\mathbf{D}$ in the following form

$$\mathbf{D}=\mathbf{M}(\mathbf{M}^\dagger\mathbf{A})^p(\det\mathbf{M})^{-p}. \qquad (17)$$

First, we check the inequalities (15). Substitution of the matrix $\mathbf{D}$ given by equation (17) into (15) yields

$$\det\mathbf{D}=\det\mathbf{M}(\det\mathbf{A})^p=\det\mathbf{M}(\omega_0^4\det\mathbf{M})^p>0,$$

$$\mathrm{tr}\mathbf{M}^\dagger\mathbf{D}=2\omega_0^{2p}\det\mathbf{M}>0.$$

To verify condition (14) we note that for $p\geq0$

$$\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M})^p=\mathrm{tr}(\mathbf{M}^\dagger\mathbf{A})^p=2(\omega_0^2\det\mathbf{M})^p, \qquad (18)$$

which can be proven by induction. Then, for $p\geq0$ we obtain after obvious transformations

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}=\frac{\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}(\mathbf{M}^\dagger\mathbf{A})^p)}{(\det\mathbf{M})^p}=2\omega_0^{2p+2}\det\mathbf{M},$$

$$\omega_0^2\mathrm{tr}\mathbf{M}^\dagger\mathbf{D}=\omega_0^2\frac{\mathrm{tr}(\mathbf{M}^\dagger\mathbf{A})^p}{(\det\mathbf{M})^{p-1}}=2\omega_0^{2p+2}\det\mathbf{M},$$

which means that condition (14) is satisfied. In the case $p<0$ we take $p=-|p|$ and represent the matrix $\mathbf{D}$ in the form

$$\mathbf{D}=\mathbf{M}(\mathbf{A}^\dagger\mathbf{M})^{|p|}(\omega_0^4\det\mathbf{M})^{-|p|}.$$

Then, with the use of (18) we obtain

$$\mathrm{tr}\mathbf{A}^\dagger\mathbf{D}=\omega_0^2\mathrm{tr}\mathbf{M}^\dagger\mathbf{D}=2\omega_0^{-2|p|+2}\det\mathbf{M}.$$

We find that for any integer p the matrix $\mathbf{D}=\mathbf{M}(\mathbf{M}^{-1}\mathbf{A})^p$ satisfies sufficient conditions of asymptotic stability (14) and (15), which required to be proved. ■

From the Lemma 3 it follows that the linear combination $\mathbf{D}=\alpha\mathbf{M}+\beta\mathbf{A}$ has a stabilizing influence on circulatory system (3), (7). Notice that formula (16) for the stabilizing velocity-dependent perturbation was found first by Walker [5]. His derivation was based on a special form of the Liapunov function. We got the same result from the study of the characteristic polynomial (4) of a general linear non-conservative system (1).

Let us find the explicit form of the *symmetric* matrices $\mathbf{D}$ realizing the perturbations stabilizing circulatory system (3), (7). Expressing Eq.(14) by means of the entries of the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{A}$ we get

$$(a_{11}-\omega_0^2m_{11})d_{22}+(a_{22}-\omega_0^2m_{22})d_{11}= \qquad (19)$$

$$=(a_{12}-\omega_0^2m_{12})d_{21}+(a_{21}-\omega_0^2m_{21})d_{12},\quad d_{12}=d_{21}.$$

Isolating the term d_{12} in Eq.(19) we can write the structure of the matrix $\mathbf{D}$ as

$$\mathbf{D}=\left\|\begin{array}{cc}d_{11}&d_{12}\\d_{21}&d_{22}\end{array}\right\|, \qquad (20)$$

$$d_{12}=d_{21}=\frac{d_{11}(\omega_0^2m_{22}-a_{22})+(\omega_0^2m_{11}-a_{11})d_{22}}{(\omega_0^2m_{12}-a_{12})+(\omega_0^2m_{21}-a_{21})}.$$

To stabilize circulatory system (3), (7) the matrices $\mathbf{D}$ with the structure (20) must also satisfy the two inequalities (15). Calculating the determinant of matrix (20) and taking into account the positiveness of $\mathrm{tr}(\mathbf{D}^\dagger\mathbf{M})$ we get the additional conditions on the entries of the symmetric matrix $\mathbf{D}$

$$\left(\frac{1-\sqrt{1-4x_1x_2}}{2x_2}\right)^2<\frac{d_{11}}{d_{22}}<\left(\frac{1+\sqrt{1-4x_1x_2}}{2x_2}\right)^2,$$

$$x_i=\frac{\omega_0^2m_{ii}-a_{ii}}{(\omega_0^2m_{12}-a_{12})+(\omega_0^2m_{21}-a_{21})},\quad i=1,2; \qquad (21)$$

$$d_{11}+d_{22}\frac{a_{11}(m_{12}+m_{21})-m_{11}(a_{12}+a_{21})}{a_{22}(m_{12}+m_{21})-m_{22}(a_{12}+a_{21})}>0. \qquad (22)$$

Therefore, symmetric matrices $\mathbf{D}$ with the structure given by Eqs.(20)–(22) realize stable perturbations of circulatory system (3), (7).

2 Two mechanical examples

2.1 The Bolotin problem

We first consider the linear non-conservative system with 2 degrees of freedom with the generalized coordinates $y_1(t)$ and $y_2(t)$ [1, 4]:

$$\frac{d^2y_1}{dt^2} + k_1\frac{dy_1}{dt} + \omega_1^2(y_1 + qb_{12}y_2) = 0,$$
$$\frac{d^2y_2}{dt^2} + k_2\frac{dy_2}{dt} + \omega_2^2(y_2 + qb_{21}y_1) = 0, \qquad (23)$$

where ω_1 and ω_2 are the eigenfrequencies of a conservative system, k_1 and k_2 are the dissipation parameters, q is the non-conservative load parameter, and b_{12}, b_{21} are the coefficients of the matrix of non-conservative positional forces. It is assumed that $b_{12}b_{21} < 0$ [1, 4]. In this problem $\mathbf{M} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix, and

$$\mathbf{D}=\begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}, \quad \mathbf{A}=\begin{bmatrix} \omega_1^2 & \omega_1^2 qb_{12} \\ \omega_2^2 qb_{21} & \omega_2^2 \end{bmatrix}. \quad (24)$$

The matrix $\mathbf{A}$ has the double eigenvalue $i\omega_0$ when the parameter q reaches its critical value

$$q_0 = \frac{|\omega_2^2 - \omega_1^2|}{2\omega_1\omega_2\sqrt{-b_{12}b_{21}}}. \qquad (25)$$

If the dissipation parameters $k_1=0$, $k_2=0$ and the load parameter q is equal to q_0 given by Eq.(25), then system (23) is circulatory and belongs to the boundary between the stability and flutter domains. According to Lemma 3 the stabilizing matrix $\mathbf{D}$ of the dissipative forces can be chosen proportional to the matrix $\mathbf{M}=\mathbf{I}$. Hence, for asymptotic stability we should at least take $k_1=k_2>0$.

A more delicate result can be obtained after calculation of the quadratic approximation of the stability domain in the plane of parameters k_1, k_2. This domain defined by inequalities (9) has the boundary, which is a part of the surface (10) approximated by Eq.(12). Calculating the necessary ingredients

$$\omega_0^2=(\omega_1^2+\omega_2^2)/2, \quad \mathrm{tr}\mathbf{D}^\dagger\mathbf{M}=k_1+k_2,$$
$$\det\mathbf{D}=k_1k_2, \quad \mathrm{tr}\mathbf{A}^\dagger\mathbf{D}=k_1\omega_1^2+k_2\omega_2^2 \qquad (26)$$

and substituting them into Eq.(12) we find

$$k_1\omega_1^2 + k_2\omega_2^2 = (k_1+k_2)\frac{\omega_1^2+\omega_2^2}{2} \pm$$
$$\pm\frac{\sqrt{\omega_1^2+\omega_2^2}}{\sqrt{2}}(k_1+k_2)\sqrt{k_1k_2}, \quad k_1, k_2 > 0. \qquad (27)$$

Seeking for the coefficient k_1 in the form $k_1=ak_2+bk_2^2+o(k_2^2)$ we get from Eq.(27)

$$a = 1, \quad b = \pm\frac{2\sqrt{2(\omega_1^2+\omega_2^2)}}{\omega_1^2-\omega_2^2} = \pm\frac{4\omega_0}{\omega_1^2-\omega_2^2}.$$

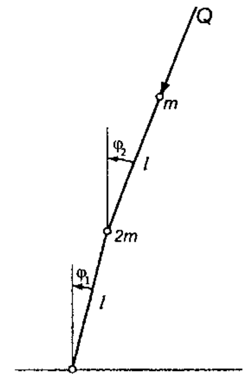

Figure 2: The Herrmann-Jong pendulum.

Finally, we arrive at the equation describing the boundary of the asymptotic stability domain

$$k_1 = k_2 \pm \frac{2\sqrt{2(\omega_1^2+\omega_2^2)}}{\omega_1^2-\omega_2^2}k_2^2 + o(k_2^2). \qquad (28)$$

Approximation (28) exactly coincides with the equation of the stability boundary obtained earlier in the work [4] from the Routh-Hurwitz criterion applied directly to system (23).

2.2 The Herrmann-Jong pendulum

Consider a double pendulum composed of two rigid weightless bars of equal length l, which carry concentrated masses $m_1=2m$, $m_2=m$. The generalized coordinates φ_1 and φ_2 are assumed to be small. A follower load Q is applied at the free end, as shown in Figure 2. At the hinges, the restoring moments $c\varphi_1+b_1d\varphi_1/dt$ and $c(\varphi_2-\varphi_1)+b_2(d\varphi_2/dt-d\varphi_1/dt)$ are induced. The linear equations of motion are

$$3ml^2\frac{d^2\varphi_1}{dt^2} + (b_1+b_2)\frac{d\varphi_1}{dt} - (Ql-2c)\varphi_1 +$$
$$+ml^2\frac{d^2\varphi_2}{dt^2} - b_2\frac{d\varphi_2}{dt} + (Ql-c)\varphi_2 = 0,$$
$$ml^2\frac{d^2\varphi_1}{dt^2} - b_2\frac{d\varphi_1}{dt} - c\varphi_1 +$$
$$+ml^2\frac{d^2\varphi_2}{dt^2} + b_2\frac{d\varphi_2}{dt} + c\varphi_2 = 0, \qquad (29)$$

where t indicates time, b_1 and b_2 are the damping coefficients and c characterizes the elastic properties of the hinges [6]. After introduction of the dimensionless quantities

$$q=\frac{Ql}{c}, \quad k_1=\frac{b_1}{\sqrt{cml^2}}, \quad k_2=\frac{b_2}{\sqrt{cml^2}}, \quad \tau=t\sqrt{\frac{c}{ml^2}}$$

we arrive at the equation in the form (1), where

$$\mathbf{M} = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix}, \quad \mathbf{D} = \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix},$$

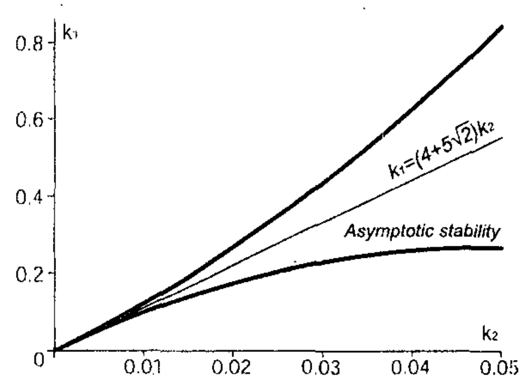

Figure 3: The domain of stabilization of the Herrmann-Jong pendulum (inside of the cusp) and the tangent cone to this domain (shown by the thin line).

$$\mathbf{A} = \begin{bmatrix} 2-q & q-1 \\ -1 & 1 \end{bmatrix}. \tag{30}$$

If the damping is absent, then the system is marginally stable for the loads $q < q_0$, where the critical load q_0 corresponds to the double eigenvalue $\lambda_0 = \pm i\omega_0$

$$\omega_0 = 2^{-1/4}, \quad q_0 = \frac{7}{2} - \sqrt{2}. \tag{31}$$

The matrix $\mathbf{D}$ of the dissipation forces in Eqs.(30) is symmetric. Let us find the structure of this matrix, which makes system (29), (31) asymptotically stable. Substituting the components of the matrices $\mathbf{A}$ and $\mathbf{M}$ evaluated at the point (31) into equation (20) we find

$$d_{12} = d_{21} = \frac{1}{23}(d_{11}(2-5\sqrt{2}) + d_{22}(17+15\sqrt{2})). \tag{32}$$

Taking into account that $d_{11} = k_1 + k_2$, $d_{22} = k_2$ we get from (32) the following condition on the dissipation parameters k_1 and k_2

$$(k_1 + k_2)(2-5\sqrt{2}) + k_2(17+15\sqrt{2}) = -k_2. \tag{33}$$

Isolating k_1 in Eq.(33) and taking into account inequalities (15) we find that

$$k_1 = (4 + 5\sqrt{2})k_2, \quad k_2 > 0 \tag{34}$$

and the stabilizing matrix of dissipative forces has the form

$$\mathbf{D} = \begin{Vmatrix} 5(1+\sqrt{2})k_2 & -k_2 \\ -k_2 & k_2 \end{Vmatrix}, \quad k_2 > 0. \tag{35}$$

Equation (34) gives the tangent cone to the asymptotic stability boundary in the plane of parameters k_1 and k_2, Figure 3. To find the more accurate approximation of the stability boundary (10) in the plane of the dissipation parameters we should just evaluate the invariants of the matrices $\mathbf{M}$, $\mathbf{D}$ and $\mathbf{A}$ at the point $q_0 = 7/2 - \sqrt{2}$ corresponding to the double eigenvalue $i\omega_0 = i2^{-1/4}$:

$$\det \mathbf{M} = 2, \quad \det \mathbf{D} = k_1 k_2,$$

$$\mathrm{tr}\mathbf{A}^{\dagger}\mathbf{D} = k_1 + k_2, \quad \mathrm{tr}\mathbf{D}^{\dagger}\mathbf{M} = k_1 + 6k_2. \tag{36}$$

Now substitute Eqs.(36) into Eq.(12) and obtain

$$k_1 + k_2 = (k_1 + 6k_2)\left(\frac{1}{\sqrt{2}} \pm \sqrt{\frac{k_1 k_2}{2\sqrt{2}}}\right). \tag{37}$$

Looking for the coefficient k_1 in the form $k_1 = ak_2 + bk_2^2 + o(k_2^2)$ finally we get from Eq.(37)

$$k_1 = (4+5\sqrt{2})k_2 \pm \sqrt{50(133+94\sqrt{2})}k_2^2 + o(k_2^2). \tag{38}$$

Asymptotic stability domain with boundaries (38) has the form of a cusp as it is shown in Figure 3. It has a singularity at the origin, so the only direction leading to the asymptotic stability domain in the plane of the dissipation parameters is given by Eq. (34). The asymptotic stability domain with boundary (38) illustrating the destabilization paradox was found first in the work [4] by the direct analysis of the characteristic equation of the Herrmann-Jong pendulum. Our approach allowed to extend the results of [4] to general non-conservative systems with two degrees of freedom.

3 Conclusions

With the use of the Leverrier–Faddejev algorithm the explicit expression for the characteristic polynomial of an $m \times m$ matrix by means of its invariants is obtained. Such a representation is found also for the characteristic polynomial of a quadratic matrix pencil. These results are applied for the detailed investigation of the stability of general non-conservative systems with 2 degrees of freedom. The necessary and sufficient conditions of asymptotic stability are obtained in terms of the matrices of the system. The geometrical interpretation of these conditions is given. The tangent cone to the asymptotic stability boundary giving simple and practical sufficient conditions of asymptotic stability is found. The structure of a stabilizing velocity-dependent perturbation of a circulatory system is established. The approximations of the asymptotic stability domain are obtained. The developed theory is compared with the results of earlier investigations and used for the study of the classical mechanical problems by V.V. Bolotin and G. Herrmann. The examples considered show the applicability and accuracy of the theoretical results obtained in the present paper.

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References

[1] Bolotin V.V. *Non-conservative Problems of the Theory of Elastic Stability.* Oxford: Pergamon Press, 1963.

[2] Seyranian A.P. and Kirillov O.N. "Bifurcation diagrams and stability boundaries of circulatory systems," *Theoretical and Applied Mechanics*, 26. pp. 135–168, 2001.

[3] Arnold V.I., *Geometrical Methods in the Theory of Ordinary Differential Equations.* New York and Berlin: Springer Verlag, 1983.

[4] Seyranian A.P. "On stabilization of non-conservative systems by dissipative forces and uncertainty of critical load," *Doklady Akademii Nauk*, 348 (3). pp. 323–326, 1996.

[5] Walker J.A., "A note on stabilizing damping configurations for linear non-conservative systems," *Int. J. Solids Struct.* 9(12). pp. 1543–1545, 1973.

[6] Herrmann G. and Jong I. C. "On nonconservative stability problems of elastic systems with slight damping." *ASME J. of Applied Mechanics* 33(1). pp. 125–133, 1966.

[7] Gantmacher F.R., *The Theory of Matrices, Volumes 1 and 2.* New York: Chelsea, 1959.

[8] Müller P.C. and Schiehlen W.O., *Linear Vibrations,* Dordrecht, The Netherlands: Martinus Nijhoff, 1985.

[9] Abramovitz M. and Stegun I.A., *Handbook of Mathematical Functions,* New York: Dover, 1972. P. 823–824.

Appendix

Consider the characteristic polynomial of a matrix $\mathbf{N} \in R^{m\times m}$

$$P_N(\lambda) \equiv (-1)^m \det(\mathbf{N} - \lambda\mathbf{I}) = \sum_{r=0}^{m} p_r\lambda^{m-r}, \quad \text{(A1)}$$

where $\mathbf{I} \in R^{m\times m}$ is the identity matrix. The coefficients p_r can be expressed through the invariants of the matrix $\mathbf{N}$ according to the *Leverrier-Faddejev algorithm* [7, 8]

$$p_0 = 1,\ rp_r = -\mathrm{tr}(\mathbf{N}\mathbf{N}_{r-1});$$

$$\mathbf{N}_0 = \mathbf{I},\ \mathbf{N}_r = \mathbf{N}\mathbf{N}_{r-1} + p_r\mathbf{I},\ r = 1\dots m. \quad \text{(A2)}$$

With the use of the notation from [9] we get the explicit expression for the coefficients p_r, $r = 1,\dots,m$

$$p_r = \frac{1}{r!}\sum_{|\alpha|_r=r}\frac{r!}{\alpha_1!\cdots\alpha_r!}\left(-\frac{\mathrm{tr}\mathbf{N}^1}{1}\right)^{\alpha_1}\cdots\left(-\frac{\mathrm{tr}\mathbf{N}^r}{r}\right)^{\alpha_r},$$

$$|\alpha|_r = \alpha_1 + 2\alpha_2 + \ldots + r\alpha_r. \quad \text{(A3)}$$

The indexes α_j are non-negative integers. Formula (A3) is proven by induction. Since $P_N(0) = \det\mathbf{N}$, the following relation is true for an $m \times m$ matrix $\mathbf{N}$

$$\sum_{\alpha_1+2\alpha_2+\ldots+r\alpha_r=r}\frac{(-\mathrm{tr}\mathbf{N}^1)^{\alpha_1}\cdots(-\mathrm{tr}\mathbf{N}^r)^{\alpha_r}}{1^{\alpha_1}\alpha_1!\cdots r^{\alpha_r}\alpha_r!} = \left\{\begin{array}{ll}\det\mathbf{N}, & r = m\\ 0, & r > m.\end{array}\right. \quad \text{(A4)}$$

Let now the matrix $\mathbf{N}$ be a 2×2 block matrix

$$\mathbf{N} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\mathbf{A} & -\mathbf{D} \end{bmatrix}, \quad \text{(A5)}$$

where the matrix with the zero entries $\mathbf{0}$, the identity matrix $\mathbf{I}$, and $\mathbf{A}, \mathbf{D}$ are real $m\times m$ matrices. The characteristic polynomial of the matrix $\mathbf{N}$ is defined by the equation

$$P_N(\lambda) = \det(\lambda^2 + \mathbf{D}\lambda + \mathbf{A}).$$

For matrix (A5) we have

$$\frac{\mathrm{tr}(\mathbf{N}^r)}{r} = \sum_{s=0}^{[\frac{r}{2}]}\frac{(r-s-1)!}{s!(r-2s)!}(-1)^{r-s}\mathrm{tr}(\mathbf{A}^s\,\mathbf{D}^{r-2s}),$$

where $\left[\frac{r}{2}\right] = \max_{z\in Z,\ 0\le z\le r/2} z$. Therefore, the characteristic polynomial of (A5) has the form

$$P_N(\lambda) = \det(\mathbf{A}+\lambda^2\mathbf{I}) + \lambda^m\det(\mathbf{D}+\lambda\mathbf{I}) - \lambda^{2m} +$$

$$+(\mathrm{tr}\mathbf{A}\mathrm{tr}\mathbf{D} - \mathrm{tr}\mathbf{A}\mathbf{D})\lambda^{2m-3} +$$

$$+\frac{\mathrm{tr}\mathbf{A}((\mathrm{tr}\mathbf{D})^2 - \mathrm{tr}\mathbf{D}^2) - 2\mathrm{tr}\mathbf{A}\mathbf{D}\mathrm{tr}\mathbf{D} + 2\mathrm{tr}(\mathbf{A}\mathbf{D}^2)}{2}\lambda^{2m-4} + \dots$$

In particular case, when $m = 2$, we have

$$P_N(\lambda) = \quad \text{(A6)}$$

$$\lambda^4 + \mathrm{tr}\mathbf{D}\lambda^3 + (\mathrm{tr}\mathbf{A} + \det\mathbf{D})\lambda^2 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{D})\lambda + \det\mathbf{A},$$

where the adjoint matrix $\mathbf{A}^\dagger$ is defined by the relation $\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}\det\mathbf{A}$, [7]. For the characteristic polynomial

$$P(\lambda) = \det(\mathbf{M}\lambda^2 + \mathbf{D}\lambda + \mathbf{A}),$$

where $\mathbf{M}$, $\mathbf{D}$, and $\mathbf{A}$ are 2×2 real matrices we obtain

$$P(\lambda) = \det\mathbf{M}\lambda^4 + \mathrm{tr}(\mathbf{D}^\dagger\mathbf{M})\lambda^3 + (\mathrm{tr}(\mathbf{A}^\dagger\mathbf{M}) +$$

$$+\det\mathbf{D})\lambda^2 + \mathrm{tr}(\mathbf{A}^\dagger\mathbf{D})\lambda + \det\mathbf{A}. \quad \text{(A7)}$$