Subcritical flutter in the acoustics of friction

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Linearized models of elastic bodies of revolution, spinning about their symmetrical axes, possess the eigenfrequency plots with respect to the rotational speed, which form a mesh with double semi-simple eigenfrequencies at the nodes. At contact with friction pads, the rotating continua, such as the singing wine glass or the squealing disc brake, start to vibrate owing to the subcritical flutter instability. In this paper, a sensitivity analysis of the spectral mesh is developed for the explicit predicting the onset of instability. The key role of the indefinite damping and non-conservative positional forces is clarified in the development and localization of the subcritical flutter. An analysis of a non-self-adjoint boundary-eigenvalue problem for a rotating circular string, constrained by a stationary load system, shows that the instability scenarios, revealed in the general two-dimensional case, are typical also in more complicated finite-dimensional and distributed systems.

1. Introduction

In 1638 Galileo Galilei remarked that 'a glass of water may be made to emit a tone merely by the friction of the fingertip upon the rim of the glass' (Galilei 2001). In 1761 Benjamin Franklin designed an 'armonica', where sound was radiated due to vibration of rotating glass bowls in frictional contact with the moistened fingers of a performer (Rossing 1994). Shortly after Rayleigh (1877) qualitatively described the onset of bending waves in the singing wine glass by the friction, applied in the circumferential direction, and pointed out the proximity of the main audible frequency of the glass to the one of the spectrum of its free vibrations, Sperry and Lanchester invented a disc brake (Kinkaid *et al.* 2003). Nowadays, disc brake squeal due to vibrations of a rotating annular plate in contact with the friction pads—in general, a sound with one dominant frequency—is the primary subject of investigations in acoustics of friction of rotating elastic bodies of revolution (Mottershead 1998; Akay 2002; Kinkaid *et al.* 2003; Ouyang *et al.* 2005).

The author of one of the first theories of squeal, Spurr (1961*a*), experimentally observed that a rotating wine glass sang when the dynamic friction coefficient was a decreasing function of the velocity (Spurr 1961*b*).

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Linearizing the system with the negative friction-velocity gradient produces an eigenvalue problem with an *indefinite* matrix of damping forces. Effectively negatively damped vibration modes may lead to complex eigenvalues with positive real parts and cause flutter instability (Freitas *et al.* 1997; Kirillov 2007*a*; Kröger *et al.* 2008; Kliem & Pommer 2008). The growth in amplitude will be limited in practice by some nonlinearity. Since the engineering design is often more concerned with if a brake may squeal and less with how loud the brake may squeal, a complex eigenvalue analysis offers for it a pragmatic approach used currently by most of production work in industry (Ouyang *et al.* 2005).

The fall in the dynamic friction coefficient with increasing velocity is among the main empirical reasons for disc brake squeal, categorized by Kinkaid *et al.* (2003). One more is *non-conservative* positional forces that first appeared in the linear models by North (1976). The binary flutter in such models happens through the coalescence of two modes according to the reversible Hopf bifurcation scenario (Kessler *et al.* 2007; Sinou & Jezequel 2007; von Wagner *et al.* 2007; Kröger *et al.* 2008). Inclusion of damping leads to the imperfect merging of modes (Hoffmann & Gaul 2003) and to the flutter through the dissipative Hopf bifurcation, which is connected to the reversible one by means of the Whitney umbrella singularity (Kirillov 2004, 2005, 2007*c*; Sinou & Jezequel 2007).

The non-conservative positional forces in the models of the frictional contact between the disc and the pads were interpreted by North (1976), Yang & Hutton (1995), Mottershead *et al.* (1997) and some other authors as tangential follower forces (Kinkaid *et al.* 2003). Despite the existing discussion of the very concept of the follower forces (Sugiyama *et al.* 1999, 2002; Elishakoff 2005), their main role is in bringing the non-potential terms, which can also be of other origin (von Wagner *et al.* 2007), into the equations of motion of brakes. The destabilizing role of non-potential positional forces in dynamical systems, including the tippe top inversion and the rising egg phenomena of rotordynamics, was emphasized recently by Krechetnikov & Marsden (2006; see also Kirillov 2007*a*,*b*; Krechetnikov & Marsden 2007; Spelsberg-Korspeter *et al.* 2008).

Historically, in the study of brake squeal, the symmetry of the disc as well as the effects of its rotation was frequently ignored. The latter in the assumption that the low rotor speed range in which squeal tends to occur does not warrant this complication (Ouyang *et al.* 2005). However, as in the case of a singing wine glass, experiments revealed the proximity of the squealing frequency and mode shape of brake's rotor for low rotational speeds to a natural frequency and corresponding mode shape of a stationary rotor (Mottershead 1998; Kinkaid *et al.* 2003; Ouyang *et al.* 2005; Giannini *et al.* 2006; Massi *et al.* 2006). Since an axially symmetric rotor possesses pairs of identical frequencies, Chan *et al.* (1994) proposed another mechanism of squeal in the classification of Kinkaid *et al.* (2003) based on the *splitting of the frequency of the doublet modes* in the symmetric disc when a friction force was applied. The splitting could lead to flutter equated to brake squeal.

Rotation also causes the doublet modes to split (Bryan 1890). The newborn pair of simple eigenvalues corresponds to the forward and backward travelling waves, which propagate along the circumferential direction (Bryan 1890; Southwell 1921). Viewed from the stationary frame, the frequency of the forward travelling wave appears to increase and that of the backward travelling wave appears to decrease, as the spin increases. Double eigenvalues thus

originate again at non-zero angular velocities, forming the nodes of the *spectral mesh* (Günther & Kirillov 2006) of the crossed frequency curves in the plane 'frequency' versus 'angular velocity'. The spectral meshes are characteristic for such rotating symmetric continua as circular strings (Schajer 1984; Yang & Hutton 1995), discs (Iwan & Stahl 1973; Hutton *et al.* 1987; Mottershead 1998), rings and cylindrical and hemispherical shells (Chang *et al.* 1996), vortex rings (Fukumoto & Hattori 2005) and a spherically symmetric α^2-dynamo of magnetohydrodynamics (Günther & Kirillov 2006).

The lowest angular velocity at which the frequency of a backward travelling wave vanishes to zero, so that the wave remains stationary in the non-rotating frame, is called *critical* (Mottershead 1998). When the speed of rotation exceeds the critical speed, the backward wave travels slower than the disc rotation speed and appears to be travelling forward (reflected wave). The effective energy of the reflected wave is negative and that of the forward and backward travelling waves is positive (MacKay & Saffman 1986). Therefore, in the *subcritical* speed region all the crossings of the frequency curves correspond to the forward and backward modes of the same signature, while in the *supercritical* speed region there exist crossings that are formed by the reflected and forward modes of opposite signature. According to Krein's theory (MacKay 1986), under Hamiltonian perturbations like the mass and stiffness constraints (Iwan & Stahl 1973), the crossings in the subcritical region veer away into *avoided crossings* (stability), while in the supercritical region the crossings of the modes of opposite signature turn into the rings of complex eigenvalues—*bubbles of instability* (MacKay & Saffman 1986)—leading to flutter known also as the 'mass and stiffness instabilities' (Iwan & Stahl 1973).

A *supercritical flutter* is important for the high-speed applications such as circular saws and computer storage devices, while in the acoustics of friction of rotating elastic bodies of revolution, a *subcritical flutter* is (un)desirable as a source of instability at low speeds. Being prohibited by Krein's theory for the Hamiltonian systems, subcritical flutter can occur, however, due to non-Hamiltonian perturbations (Bridges 1997). In the 1990s, subcritical flutter was detected by numerical approaches in the new models of disc brakes that incorporated gyroscopic and centripetal effects and accommodated more than one squeal mechanism through the splitting of the doublet modes of a disc by dissipative and non-conservative perturbations coming from the negative friction-velocity gradient and frictional follower load. The models include both the case when the pointwise or distributed friction pads are rotated around a stationary disc, affecting a point or a sector of it, and when the disc rotates past the stationary friction pads (see Chan *et al.* 1994; Mottershead *et al.* 1997; Xiong *et al.* 2002; Kinkaid *et al.* 2003; Ouyang *et al.* 2005 and references therein). Subcritical parametric resonance in the former of the two dual descriptions (Shapiro 2001) corresponds to the subcritical flutter in the latter.

In this paper, we propose a sensitivity analysis based on the perturbation theory of multiple eigenvalues of non-self-adjoint operators (Kirillov & Seyranian 2004, 2005; Kirillov *et al.* 2005; Günther & Kirillov 2006), which is an efficient tool for investigation of the subcritical flutter in both the finite-dimensional and distributed models. Instead of deriving the particular operators of dissipative and circulatory forces by accurate modelling of the frictional contact and then studying their effect on the spectrum and stability,

we solve an inverse problem. Assuming *a priori* only the existence of distinct squeal frequencies close to the double eigenfrequencies of the non-rotating body, we find the structure of the dissipative and non-conservative operators whose action causes flutter in the subcritical region near the nodes of the spectral mesh. We describe analytically the movement of eigenvalues and the deformation of the spectral mesh. Using these data, we approximate the stability domain in the space of system's parameters.

Confirming an empirical duality—supercritical flutter due to indefiniteness of the matrix of potential forces and subcritical flutter due to indefiniteness of the matrix of damping forces—we come to new qualitative conclusions. A discovered singularity of the stability boundary allows for the combinations of dissipative and non-conservative positional forces yielding the subcritical flutter instability in the vicinity of the nodes of the spectral mesh even in the case, when the damping matrix is positive definite with some of its eigenvalues close to zero. The vanishing and negative eigenvalues of the damping matrix encourage the development of the subcritical flutter while zero eigenvalues of the matrix of non-conservative positional forces suppress it. The proposed approach provides guidance to the classification of dissipative and non-conservative perturbations by their ability to cause the subcritical flutter, which is helpful in checking and correcting particular models of disc brakes and other rotating elastic bodies of revolution in frictional contact.

2. The spectral mesh of a two-dimensional gyroscopic system

Consider a non-dimensional equation of an autonomous non-conservative system

$$\ddot{\boldsymbol{x}} + (2\varOmega\boldsymbol{G} + \delta\boldsymbol{D})\dot{\boldsymbol{x}} + ((\beta^2 - \varOmega^2)\boldsymbol{I} + \kappa\boldsymbol{K} + \nu\boldsymbol{N})\boldsymbol{x} = 0, \tag{2.1}$$

where a dot over a symbol denotes time differentiation $\boldsymbol{x} \in \mathbb{R}^2$ and $\boldsymbol{I}$ is the identity matrix. The real matrices $\boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}}$, $\boldsymbol{G} = -\boldsymbol{G}^{\mathrm{T}}$, $\boldsymbol{K} = \boldsymbol{K}^{\mathrm{T}}$ and $\boldsymbol{N} = -\boldsymbol{N}^{\mathrm{T}}$ are related to dissipative (damping), gyroscopic, potential and non-conservative positional (circulatory) forces with magnitudes controlled by the scaling factors δ, $\varOmega$, κ and ν, respectively; $\beta > 0$ is the frequency of free vibrations of the potential system when $\delta = \varOmega = \kappa = \nu = 0$. Without loss of generality, we assume $\det \boldsymbol{G} = \det \boldsymbol{N} = 1$. Equation (2.1) originates as a two-mode approximation of the models of rotating bodies of revolution in frictional contact after their linearization and discretization (Nagata & Namachchivaya 1998; Spelsberg-Korspeter *et al*. submitted). It appeared recently in the study of the Benjamin–Feir instability by Bridges & Dias (2007).

Separating time by setting $\boldsymbol{x}(t) = \boldsymbol{u}\exp(\lambda t)$ we arrive at the eigenvalue problem

$$\boldsymbol{L}\boldsymbol{u} = 0, \quad \boldsymbol{L} = \boldsymbol{I}\lambda^2 + (2\varOmega\boldsymbol{G} + \delta\boldsymbol{D})\lambda + (\beta^2 - \varOmega^2)\boldsymbol{I} + \kappa\boldsymbol{K} + \nu\boldsymbol{N}. \tag{2.2}$$

The operator $\boldsymbol{L}_0(\varOmega) = \boldsymbol{I}\lambda^2 + 2\lambda\varOmega\boldsymbol{G} + (\beta^2 - \varOmega^2)\boldsymbol{I}$ has four eigenvalues (Nagata & Namachchivaya 1998; Hryniv & Lancaster 2001),

$$\lambda_{\mathrm{p}}^{\pm} = \mathrm{i}\beta \pm \mathrm{i}\varOmega, \qquad \lambda_n^{\pm} = -\mathrm{i}\beta \pm \mathrm{i}\varOmega, \tag{2.3}$$
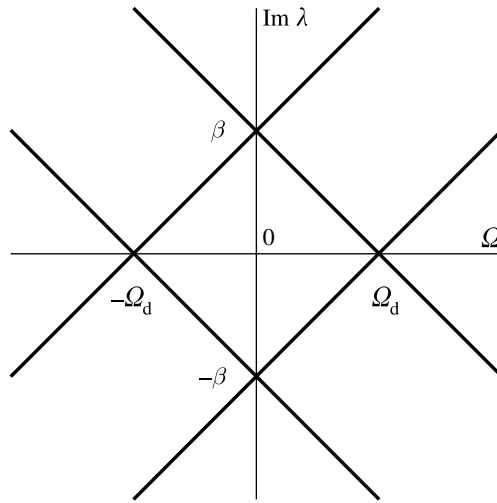
Figure 1. The spectral mesh of system (2.1) when $\delta=\nu=\kappa=0$.

forming the spectral mesh in the plane $(\Omega, \text{Im } \lambda)$ (figure 1). Two nodes of the mesh at $\Omega=\Omega_0=0$ in the subcritical interval $|\Omega|<\Omega_d=\beta$ correspond to the double semi-simple eigenvalues $\lambda=\pm i\beta$. The eigenvalue $i\beta$ has the following two orthogonal eigenvectors:

$$\boldsymbol{u}_1 = \frac{1}{\sqrt{2\beta}}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad \boldsymbol{u}_2 = \frac{1}{\sqrt{2\beta}}\begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{2.4}$$

At the other two nodes at $\Omega=\pm\Omega_d$ double semi-simple eigenvalues are zero.

Under perturbation $\Omega=\Omega_0+\Delta\Omega$, the eigenvalue $i\beta$ into two simple ones bifurcates according to the asymptotic formula (Kirillov *et al.* 2005)

$$\lambda_p^{\pm} = i\beta + i\Delta\Omega\frac{f_{11}+f_{22}}{2}\pm i\Delta\Omega\sqrt{\frac{(f_{11}-f_{22})^2}{4}+f_{12}f_{21}}, \tag{2.5}$$

where the quantities f_{jk} are

$$f_{jk} = \boldsymbol{u}_k^{\mathrm{T}}\frac{\partial \boldsymbol{L}_0(\Omega)}{\partial\Omega}\boldsymbol{u}_j\bigg|_{\Omega=0,\,\lambda=i\beta} = 2i\beta\boldsymbol{u}_k^{\mathrm{T}}\boldsymbol{G}\boldsymbol{u}_j. \tag{2.6}$$

With $\boldsymbol{G}=-\boldsymbol{G}^{\mathrm{T}}$ formula (2.5) yields (2.3), because $f_{jj}=0$ and $f_{12}=-f_{21}=i$.

A general approach to establishing conditions for existence of the spectral mesh in multi-dimensional Hamiltonian systems was proposed by Dellnitz *et al.* (1992) and Dellnitz & Melbourne (1994). Although the complete investigation of the spectral mesh and its deformation under both Hamiltonian and non-Hamiltonian perturbations in system (2.1) with arbitrary number of degrees of freedom would be very desirable for applications, a restriction to two dimensions is justified for demonstrating the basic ideas of our theory. On the other hand, two-dimensional models are widely employed in acoustics of friction (von Wagner *et al.* 2007), while our perturbative approach does not depend on the number of degrees of freedom.

3. Deformation of the spectral mesh

Consider a general perturbation of the gyroscopic system $\boldsymbol{L}_0(\varOmega) + \Delta \boldsymbol{L}(\varOmega)$, assuming that the size of the perturbation $\Delta \boldsymbol{L}(\varOmega) = \delta \lambda \boldsymbol{D} + \kappa \boldsymbol{K} + \nu \boldsymbol{N} \sim \varepsilon$ is small, where $\varepsilon = \|\Delta \boldsymbol{L}(0)\|$ is the Frobenius norm of the perturbation at $\varOmega = 0$. For small $\varOmega$ and ε perturbation of $\lambda = \mathrm{i}\beta$ is described by the asymptotic formula (Kirillov *et al.* 2005)

$$\lambda_{\mathrm{p}}^{\pm} = \mathrm{i}\beta + \mathrm{i}\varOmega \frac{f_{11} + f_{22}}{2} + \mathrm{i}\frac{\epsilon_{11} + \epsilon_{22}}{2}$$

$$\pm \mathrm{i}\sqrt{\frac{(\varOmega(f_{11} - f_{22}) + \epsilon_{11} - \epsilon_{22})^2}{4} + (\varOmega f_{12} + \epsilon_{12})(\varOmega f_{21} + \epsilon_{21})}, \qquad (3.1)$$

where f_{jk} are given by (2.6) and ϵ_{jk} are small complex numbers of order ε,

$$\epsilon_{jk} = \boldsymbol{u}_k^{\mathrm{T}} \Delta \boldsymbol{L}(0) \boldsymbol{u}_j = \mathrm{i}\beta \delta \boldsymbol{u}_k^{\mathrm{T}} \boldsymbol{D} \boldsymbol{u}_j + \kappa \boldsymbol{u}_k^{\mathrm{T}} \boldsymbol{K} \boldsymbol{u}_j + \nu \boldsymbol{u}_k^{\mathrm{T}} \boldsymbol{N} \boldsymbol{u}_j. \qquad (3.2)$$

With the vectors (2.4) we obtain

$$\mathrm{Re}\,\lambda = -\frac{\mu_1 + \mu_2}{4}\delta \pm \sqrt{\frac{|c| + \mathrm{Re}\,c}{2}}, \quad \mathrm{Im}\,\lambda = \beta + \frac{\rho_1 + \rho_2}{4\beta}\kappa \pm \sqrt{\frac{|c| - \mathrm{Re}\,c}{2}}, \quad (3.3)$$

$$\left.\begin{aligned} \mathrm{Re}\,c &= \left(\frac{\mu_1 - \mu_2}{4}\right)^2 \delta^2 - \left(\frac{\rho_1 - \rho_2}{4\beta}\right)^2 \kappa^2 - \varOmega^2 + \frac{\nu^2}{4\beta^2}, \\ \mathrm{Im}\,c &= \frac{\varOmega\nu}{\beta} - \delta\kappa \frac{2\,\mathrm{tr}\,\boldsymbol{K}\boldsymbol{D} - \mathrm{tr}\,\boldsymbol{K}\,\mathrm{tr}\,\boldsymbol{D}}{8\beta}, \end{aligned}\right\} \qquad (3.4)$$

where the eigenvalues $\mu_{1,2}$ and $\rho_{1,2}$ of the matrices $\boldsymbol{D}$ and $\boldsymbol{K}$ satisfy the equations

$$\mu^2 - \mu\,\mathrm{tr}\,\boldsymbol{D} + \det\boldsymbol{D} = 0, \qquad \rho^2 - \rho\,\mathrm{tr}\,\boldsymbol{K} + \det\boldsymbol{K} = 0. \qquad (3.5)$$

Formulae (3.3) take into account the forces of all types and explicitly describe the perturbed spectrum by means of the eigenelements and the derivatives of the operator with respect to the parameters, calculated solely at the nodes of the spectral mesh. This is more efficient for describing the deformation of the spectral mesh, e.g. the veering and merging of eigenvalue branches (Leissa 1974; Perkins & Mote 1986), than the sensitivity analysis of simple eigenvalues by Yang & Hutton (1995), Vidoli & Vestroni (2005) and Huang *et al.* (2007).

(*a*) *Conservative deformation of the spectral mesh*

A deformation of the spectral mesh with $\delta = \nu = 0$ does not shift the eigenvalues from the imaginary axis, preserving the marginal stability. From expressions (3.3) and (3.4), we find that near the node $(0, \beta)$ in the plane $(\varOmega, \mathrm{Im}\,\lambda)$

$$\left(\mathrm{Im}\,\lambda - \beta - \frac{\rho_1 + \rho_2}{4\beta}\kappa\right)^2 - \varOmega^2 = \left(\frac{\rho_1 - \rho_2}{4\beta}\right)^2 \kappa^2, \quad \mathrm{Re}\,\lambda = 0. \qquad (3.6)$$
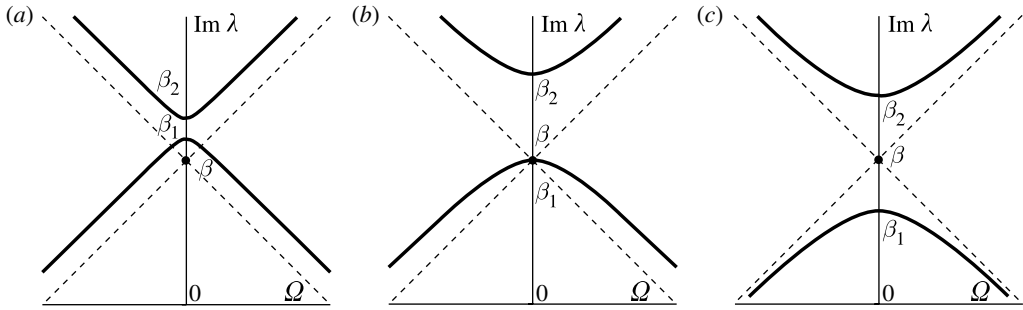
Figure 2. Conservative deformation of the spectral mesh ($\kappa > 0$): $\boldsymbol{K}$ is (*a*) positive-definite, (*b*) positive semi-definite, and (*c*) indefinite.

For $\kappa \neq 0$, equation (3.6) describes a hyperbola with the asymptotes

$$\text{Im } \lambda = \beta + \frac{\rho_1 + \rho_2}{4\beta} \kappa \pm \Omega. \tag{3.7}$$

The asymptotes cross each other above the node $(0, \beta)$ of the non-deformed spectral mesh for tr $\boldsymbol{K} > 0$, exactly at the node for $\rho = -\rho_2$, and below the node for tr $\boldsymbol{K} < 0$. The branches of the hyperbola intersect the axis $\Omega = 0$ at the points

$$\beta_1 = \beta + \frac{\rho_1}{2\beta} \kappa, \qquad \beta_2 = \beta + \frac{\rho_2}{2\beta} \kappa. \tag{3.8}$$

If the eigenvalues $\rho_{1,2}$ have the same sign, the intersection points are above the node for $\boldsymbol{K} > 0$ and below it for $\boldsymbol{K} < 0$ (figure 2*a*). When one of the eigenvalues $\rho_{1,2}$ is zero, which implies semi-definiteness of the matrix $\boldsymbol{K}$, one of the branches of the hyperbola passes through the node. The other one crosses the axis $\Omega = 0$ above the node, if $\boldsymbol{K} \geq 0$ or below it, if $\boldsymbol{K} \leq 0$ (figure 2*b*). If $\boldsymbol{K}$ is indefinite, one of the points $\beta_{1,2}$ is located above the node and another one below it (figure 2*c*).

(*b*) *Creating and activating the latent sources of instability by dissipation*

Assuming $\nu = \kappa = 0$ in expressions (3.3) and (3.4) we find that

$$\left(\text{Re } \lambda + \frac{\mu_1 + \mu_2}{4} \delta\right)^2 + \Omega^2 = \frac{(\mu_1 - \mu_2)^2}{16} \delta^2, \quad \text{Im } \lambda = \beta \quad \text{for} \quad \text{Re } c > 0, \quad (3.9)$$

$$\Omega^2 - (\text{Im } \lambda - \beta)^2 = \frac{(\mu_1 - \mu_2)^2}{16} \delta^2, \quad \text{Re } \lambda = -\frac{\mu_1 + \mu_2}{4} \delta \quad \text{for} \quad \text{Re } c < 0. \quad (3.10)$$

In the three-dimensional space (Ω, Im λ, Re λ), the circle of complex eigenvalues (3.9) belongs to the plane Im $\lambda = \beta$, while the hyperbola (3.10) lies in the plane Re $\lambda = -\delta(\mu_1 + \mu_2)/4$, as shown in figures 3*a*,*c* and 4*a*,*c*.

According to (3.9) the radius of the bubble of instability r_b and the distance d_b of its centre from the plane Re $\lambda = 0$ are defined by the eigenvalues $\mu_{1,2}$ of $\boldsymbol{D}$,

$$r_b = \frac{|(\mu_1 - \mu_2)\delta|}{4}, \qquad d_b = \frac{|(\mu_1 + \mu_2)\delta|}{4}. \tag{3.11}$$

The bubble of complex eigenvalues and hence the branches of the adjacent hyperbola (3.10) are 'submerged' under the surface Re $\lambda = 0$, when the conditions $d_b \geq r_b$ and δtr $\boldsymbol{D} > 0$ are fulfilled, yielding the positive (semi-)definite matrix $\delta \boldsymbol{D}$
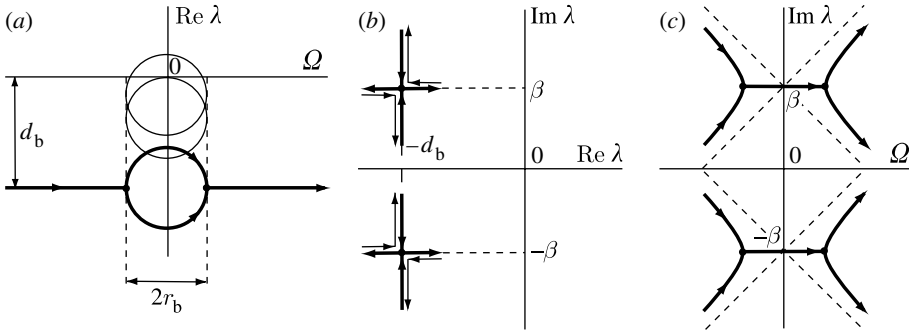
Figure 3. Origination of a latent source of subcritical flutter instability in the presence of full dissipation: (a) submerged bubble of instability, (b) coalescence of eigenvalues in the complex plane at two exceptional points, and (c) hyperbolic trajectories of imaginary parts.

of (pervasive) full damping. In the complex plane, the eigenvalues move with the variation of Ω along the lines $\mathrm{Re}\,\lambda = -d_{\mathrm{b}}$ until they meet at the junction of the bubble of instability (3.9) with the hyperbola (3.10),

$$\mathrm{Im}\,\lambda = \beta, \qquad \mathrm{Re}\,\lambda = -\delta(\mu_1 + \mu_2)/4, \qquad \Omega = \pm\delta(\mu_1 - \mu_2)/4, \qquad (3.12)$$

and form the double eigenvalue with the Jordan chain of two generalized eigenvectors (exceptional point). With further increase in Ω the eigenvalues split in the orthogonal direction, never crossing the imaginary axis (figure 3b). A similar process of origination of two exceptional points from the semi-simple eigenvalue (diabolical point) in Hermitian matrices under complex perturbation with application to crystal optics was described by Berry & Dennis (2003) and Kirillov *et al.* (2005).

For the phenomenon of squeal, it is important that the dissipation-induced bubble of complex eigenvalues, localized in the subcritical interval $|\Omega| < \Omega_{\mathrm{d}}$, is a latent source of unstable modes with the frequencies close to the repeated eigenfrequency $\mathrm{Im}\,\lambda = \beta$ of the non-rotating system. In the absence of circulatory forces the radius of the bubble of instability (3.11) is greater than the depth of its submersion under the surface $\mathrm{Re}\,\lambda = 0$, only if the eigenvalues $\mu_{1,2}$ of $\boldsymbol{D}$ have different signs. The eigenvalues of the emerged bubble have positive real parts in the range $\Omega^2 < \Omega_{\mathrm{cr}}^2$, where $\Omega_{\mathrm{cr}} = (\delta/2)\sqrt{-\det\boldsymbol{D}}$, confirming that the negative friction-velocity gradient as a source of indefinite damping can be a reason for subcritical flutter and squeal.

The sector-shaped domain of asymptotic stability of system (2.1) with indefinite damping is defined by the constraints $\delta\,\mathrm{tr}\,\boldsymbol{D} > 0$ and $\Omega^2 > \Omega_{\mathrm{cr}}^2$. Owing to the singularity at the origin in the plane (δ, Ω), an unstable system with indefinite damping can be stabilized by sufficiently strong gyroscopic forces, as shown by the dashed line in figure 5a. With the increase in $\det\boldsymbol{D}$ the stability domain gets wider and for $\det\boldsymbol{D} > 0$ it is defined by the condition $\delta\,\mathrm{tr}\,\boldsymbol{D} > 0$ (figure 5c). At $\det\boldsymbol{D} = 0$, the line $\Omega = 0$ does not belong to the domain of asymptotic stability (figure 5b). Changing the matrix $\delta\boldsymbol{D}$ from positive definite to indefinite triggers the state of the bubble of instability from the latent ($\mathrm{Re}\,\lambda < 0$) to the active one ($\mathrm{Re}\,\lambda > 0$) (figure 4a,c).
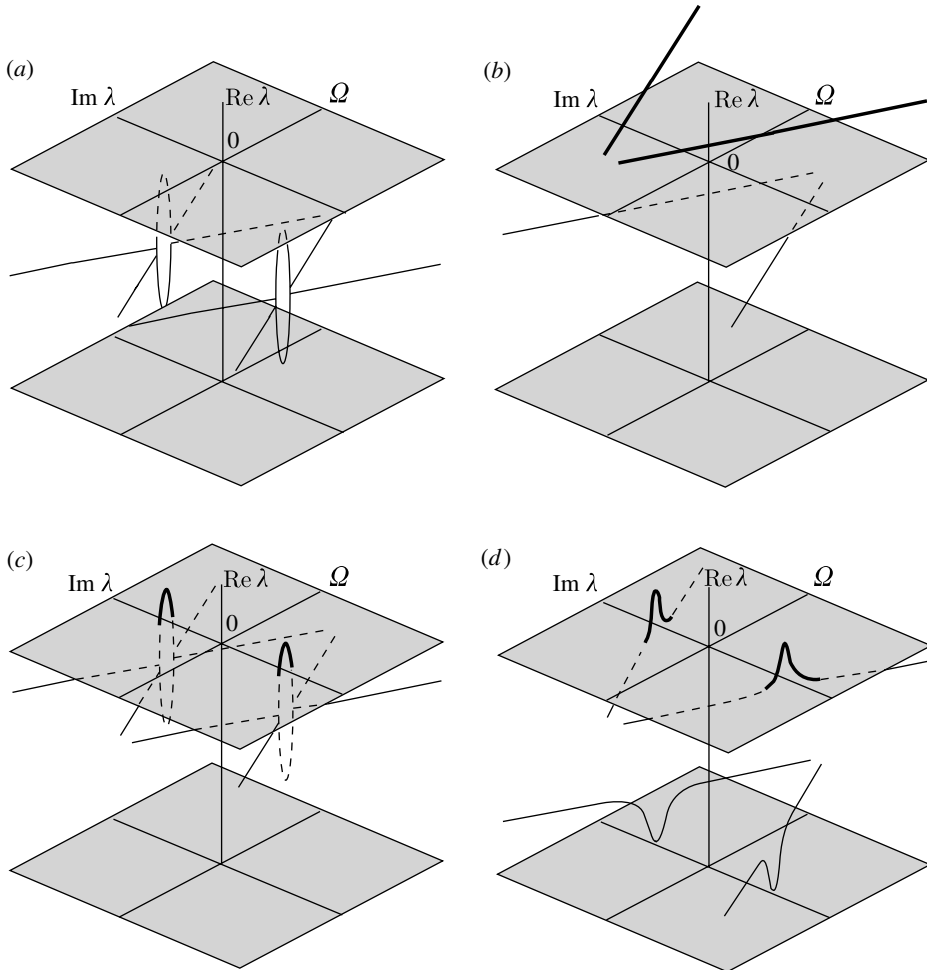
Figure 4. The mechanism of subcritical flutter (bold lines): (*a*) the ring (bubble) of complex eigenvalues under the surface Re $\lambda = 0$—a latent source of instability created by dissipation with det $\boldsymbol{D} \geq 0$, (*b*) repulsion of eigenvalue branches of the spectral mesh by non-conservative perturbations, (*c*) emersion of the bubble of instability due to indefinite damping with det $\boldsymbol{D} < 0$, and (*d*) collapse of the bubble of instability and immersion and emersion of its parts due to combined action of dissipative and non-conservative positional forces.

(*c*) *Activating the bubble of instability by non-conservative positional forces*

In the absence of dissipation, the non-conservative positional forces destroy the marginal stability of gyroscopic systems (Lakhadanov 1975). Assuming $\delta = \kappa = 0$ in (3.3) and (3.4), we find that the eigenvalues of the branches $\pm(\mathrm{i}\beta + \mathrm{i}\Omega)$ of the spectral mesh get positive real parts due to a non-conservative perturbation

$$\lambda_{\mathrm{p}}^{\pm} = \mathrm{i}\beta \pm \mathrm{i}\Omega \pm \frac{\nu}{2\beta}, \qquad \lambda_{n}^{\pm} = -\mathrm{i}\beta \pm \mathrm{i}\Omega \mp \frac{\nu}{2\beta}. \qquad (3.13)$$

In contrast to the effect of indefinite damping, the circulatory forces destabilize one of the two modes at every Ω (figure 4*b*). In order to localize the instability in the vicinity of the nodes, a combination of circulatory and dissipative forces is required.
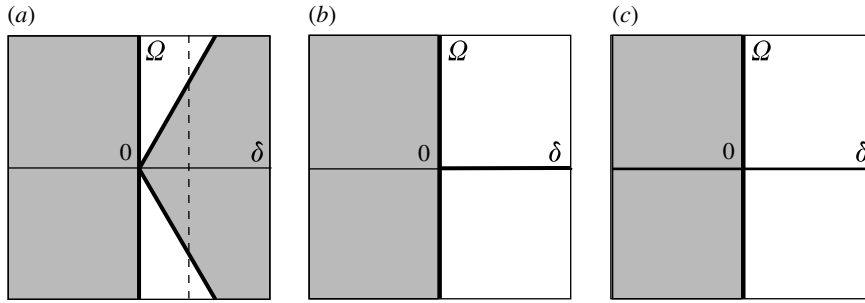
Figure 5. Approximation to the domain of asymptotic stability (white portion) and its boundary (bold lines) for the dissipatively perturbed gyroscopic system with tr $\boldsymbol{D} > 0$: (*a*) det $\boldsymbol{D} = 0$, (*b*) det $\boldsymbol{D} > 0$, and (*c*) det $\boldsymbol{D} > 0$.
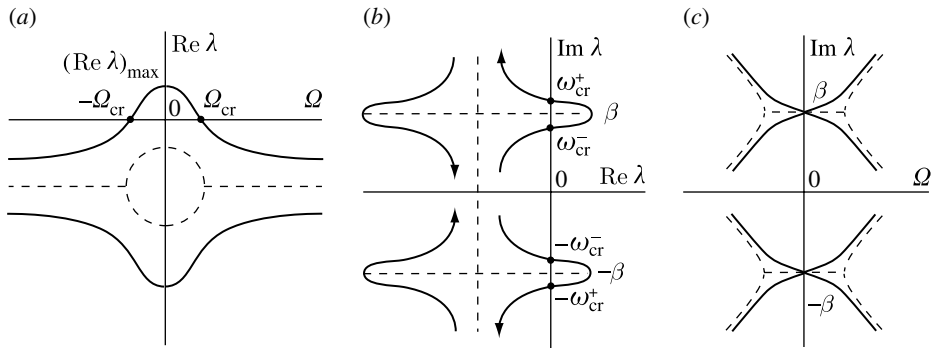

Figure 6. Subcritical flutter caused by the dissipative and circulatory forces: (*a*) collapse and emersion of the bubble of instability, (*b*) excursions of eigenvalues to the r.h.s. of the complex plane when Ω increases, and (*c*) crossing of the imaginary parts.

With $\kappa = 0$ in (3.3) and (3.4), we describe the trajectories of the eigenvalues in the complex plane in the presence of dissipative and non-conservative perturbations

$$\left(\operatorname{Re} \lambda + \frac{\operatorname{tr} \boldsymbol{D}}{4} \delta \right) (\operatorname{Im} \lambda - \beta) = \frac{\Omega \nu}{2\beta}. \tag{3.14}$$

Circulatory forces destroy the merging of modes shown in figure 3, causing the eigenvalues to move along the separated trajectories. According to (3.13) and (3.14), the eigenvalues with $|\operatorname{Im} \lambda|$ increasing with an increase in $|\Omega|$ move closer to the imaginary axis than the others, as shown in figure 6*b*. The non-conservative perturbation separates the bubble of instability and the adjacent hyperbolic eigenvalue branches into two non-intersecting curves in the space $(\Omega, \operatorname{Im} \lambda, \operatorname{Re} \lambda)$ (figure 4*d*). The remnants of the original bubble of instability yield subcritical flutter at a frequency $\omega_{\mathrm{cr}}^- < \omega < \omega_{\mathrm{cr}}^+$ with the gyroscopic parameter in the range $\Omega^2 < \Omega_{\mathrm{cr}}^2$, where

$$\Omega_{\mathrm{cr}} = \delta \frac{\operatorname{tr} \boldsymbol{D}}{4} \sqrt{-\frac{\nu^2 - \delta^2 \beta^2 \det \boldsymbol{D}}{\nu^2 - \delta^2 \beta^2 (\operatorname{tr} \boldsymbol{D}/2)^2}}, \qquad \omega_{\mathrm{cr}}^{\pm} = \beta \pm \frac{\nu}{2\beta} \sqrt{-\frac{\nu^2 - \delta^2 \beta^2 \det \boldsymbol{D}}{\nu^2 - \delta^2 \beta^2 (\operatorname{tr} \boldsymbol{D}/2)^2}}. \tag{3.15}$$
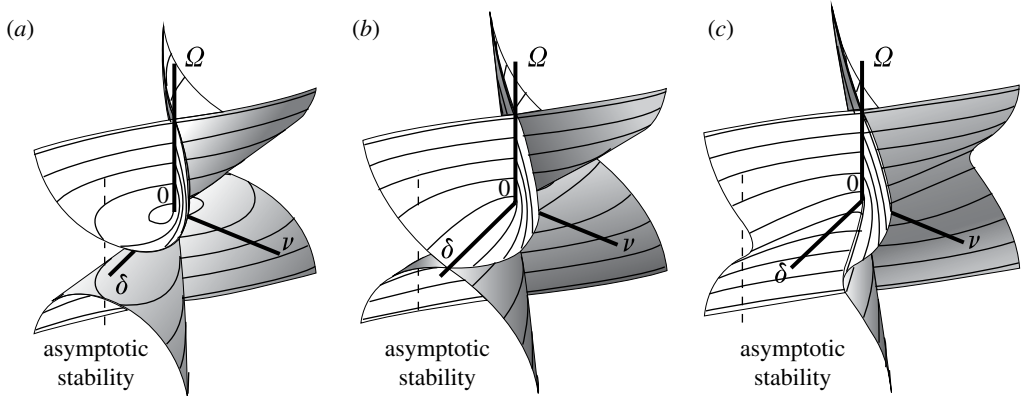
Figure 7. Domains of asymptotic stability in the space δ, ν and Ω for different types of damping: (a) det $\boldsymbol{D} < 0$, (b) det $\boldsymbol{D} = 0$, and (c) det $\boldsymbol{D} > 0$.

Hence, in the presence of the non-conservative positional forces the excursions of eigenvalues to the r.h.s. of the complex plane shown in figure 6b are possible, even if the dissipation is full (det $\boldsymbol{D} > 0$).

The first of equations (3.15) approximates the stability boundary in the space of the parameters δ, ν and Ω. Extracting ν in (3.15) yields

$$\nu = \pm \, \delta\beta \, \mathrm{tr} \, \boldsymbol{D} \sqrt{\frac{\delta^2 \, \det \, \boldsymbol{D} + 4\Omega^2}{\delta^2 (\mathrm{tr} \, \boldsymbol{D})^2 + 16\Omega^2}}. \tag{3.16}$$

If det $\boldsymbol{D} \geq 0$ and Ω is fixed, the formula (3.16) describes two independent curves in the plane (δ, ν), intersecting with each other at the origin along the straight lines

$$\nu = \pm \frac{\beta \, \mathrm{tr} \, \boldsymbol{D}}{2} \delta. \tag{3.17}$$

For det $\boldsymbol{D} < 0$ equation (3.16) describes in the plane (δ, ν) two branches of a closed loop, self-intersecting at the origin with the tangents (3.17). In the space of the three parameters, the surface (3.16) is a cone with the '8'-shaped loop in a cross section (figure 7a). Asymptotic stability is inside the two of four pockets of the cone, selected by the inequality $\delta \mathrm{tr} \, \boldsymbol{D} > 0$. The singularity at the origin is the degeneration of a more general configuration found by Kirillov (2007a).

The domain of asymptotic stability bifurcates with the change of sign of det $\boldsymbol{D}$. In the case of indefinite damping, an instability gap exists due to the singularity at the origin (figure 7a). For det $\boldsymbol{D} = 0$ the gap vanishes in the direction $\nu = 0$ (figure 7b). Despite the full dissipation with det $\boldsymbol{D} > 0$ unfolds the singularity, the memory about the instability gap is preserved in the two folds of the stability boundary with the locally strong curvature (figure 7c). When both $\mu_1 > 0$ and $\mu_2 > 0$, the folds are more pronounced, if one of the eigenvalues is close to zero. If the eigenvalues $\mu_{1,2}$ have different signs, subcritical flutter is possible for any combination of δ and ν including the case when the non-conservative positional forces are absent ($\nu = 0$).

Independently on the structure of the matrix $\boldsymbol{D}$, the primary role of dissipation is the creation of the bubble of instability. It is submerged below the surface Re $\lambda = 0$ in the space (Ω, Im λ, Re λ) in the case of full dissipation and
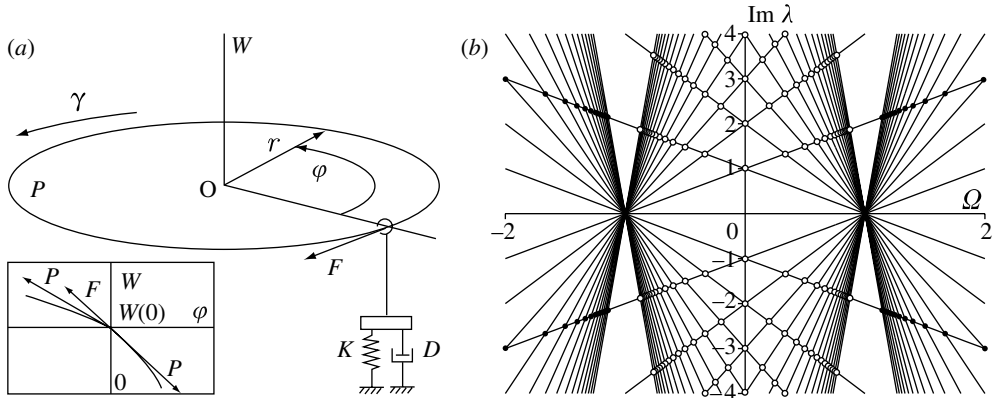
Figure 8. (*a*) A rotating circular string and (*b*) its 'keyboard' constituted by the nodes (marked by open and filled circles) of the spectral mesh (only 30 modes are shown).

partially lies in the domain $\mathrm{Re}\,\lambda > 0$ when damping is indefinite. Non-conservative positional forces destroy the bubble into two branches and shift one of them to the region of positive real parts even in the case of full dissipation. Since the branch remembers the existence of the bubble, the subcritical flutter is developing near the nodes of the spectral mesh.

4. Example: a rotating circular string

The perturbative approach of §3, modified along the lines of the work (Kirillov & Seyranian 2004), is applicable to the *non-discretized* boundary-eigenvalue problems, associated with the rotating strings, rings, discs and shells in frictional contact for a wide class of available boundary conditions. We note, however, that the correct formulation of the boundary conditions for such problems is a delicate question, which is not resolved yet in full in the existing literature (see, for example, Spelsberg-Korspeter *et al.* 2008, submitted).

The eigenvalue behaviour predicted by the analysis of the general two-dimensional system of §3 was already observed by Hutton *et al.* (1987), Yang & Hutton (1995) and Xiong *et al.* (2002), who studied a rotating disc and a rotating circular string in a pointwise contact with the stationary load systems.

For simplicity, following Yang & Hutton (1995), we consider a circular string of displacement $W(\varphi, \tau)$, radius r and mass per unit length ρ that rotates with the speed γ and passes at $\varphi = 0$ through a massless eyelet generating a constant frictional follower force F on the string, as shown in figure 8. The circumferential tension P in the string is assumed to be constant; the stiffness of the spring supporting the eyelet is K and the damping coefficient of the viscous damper is D; the velocity of the string in the φ direction has constant value γr. This a somewhat artificial system contains, however, the fundamental physics of interest, i.e. the interaction of rotating flexible medium with a stationary constraint in which the inertial, gyroscopic and centripetal acceleration effects, together with the stiffness effects of the medium, are in dynamic equilibrium with the forces generated by the constraint.

With the non-dimensional variables and parameters

$$t = \frac{\tau}{r}\sqrt{\frac{P}{\rho}}, \quad w = \frac{W}{r}, \quad \Omega = \gamma r \sqrt{\frac{\rho}{P}}, \quad k = \frac{Kr}{P}, \quad \mu = \frac{F}{P}, \quad d = \frac{D}{\sqrt{\rho P}}, \quad (4.1)$$

the substitution of $w(\varphi, t) = u(\varphi)\exp(\lambda t)$ into the governing equation and boundary conditions yields the boundary-eigenvalue problem (Yang & Hutton 1995)

$$Lu = \lambda^2 u + 2\Omega\lambda u' - (1 - \Omega^2)u'' = 0, \quad (4.2)$$

$$u(0) - u(2\pi) = 0, \qquad u'(0) - u'(2\pi) = \frac{\lambda d + k}{1 - \Omega^2}u(0) + \frac{\mu}{1 - \Omega^2}u'(0), \quad (4.3)$$

where $' = \partial_\varphi$. The non-self-adjoint boundary-eigenvalue problem (4.2) and (4.3) depends on the speed of rotation (Ω), and damping (d), stiffness (k) and friction (μ) coefficients of the constraint. For comparing the eigenvalue movement with that of §3, some artificialness of the term, corresponding to the non-conservative positional forces, in the second of the boundary conditions (4.3), discussed by Yang & Hutton (1995), Tian & Hutton (1999) and O'Reilly & Varadi (2004), is less important than the presence in (4.3) of all types of the perturbations involved in the equation (2.1).

Since the unconstrained rotating string is a gyroscopic system, the eigenfunctions of the adjoint eigenvalue problems, corresponding to a purely imaginary eigenvalue λ, coincide. With $u = C_1\exp(\varphi\lambda/(1 - \Omega)) + C_2\exp(-\varphi\lambda/(1 + \Omega))$ assumed as a solution of (4.2) in (4.3), we find the characteristic equation

$$8\lambda \sin\frac{\pi\lambda}{\mathrm{i}(1 - \Omega)} \sin\frac{\pi\lambda}{\mathrm{i}(1 + \Omega)} \frac{\mathrm{e}^{(-2\pi\lambda\Omega)/(\Omega^2 - 1)}}{\Omega^2 - 1} = 0. \quad (4.4)$$

The eigenvalues of the eigenvalue problem (4.2), (4.3), formed by the roots of (4.4)

$$\lambda_n^+ = \mathrm{i}n(1 + \Omega), \qquad \lambda_n^- = \mathrm{i}n(1 - \Omega), \qquad n \in \mathbb{Z}, \quad (4.5)$$

have the eigenfunctions $u_n^\pm = \cos(n\varphi) \mp \mathrm{i}\sin(n\varphi)$. The spectral mesh (4.5) in the plane (Ω, Im λ) is shown in figure 8.

Two eigenvalue branches $\lambda_n^\varepsilon = \mathrm{i}n(1 + \varepsilon\Omega)$ and $\lambda_m^\delta = \mathrm{i}m(1 + \delta\Omega)$, where $\varepsilon, \delta = \pm 1$, intersect each other at the node ($\Omega_{mn}^{\varepsilon\delta}, \lambda_{mn}^{\varepsilon\delta}$) with

$$\Omega_{mn}^{\varepsilon\delta} = \frac{n - m}{m\delta - n\varepsilon}, \qquad \lambda_{mn}^{\varepsilon\delta} = \frac{\mathrm{i}nm(\delta - \varepsilon)}{m\delta - n\varepsilon}, \quad (4.6)$$

where the double eigenvalue $\lambda_{mn}^{\varepsilon\delta}$ has two linearly independent eigenfunctions

$$u_n^\varepsilon = \cos(n\varphi) - \varepsilon\mathrm{i}\sin(n\varphi), \quad u_m^\delta = \cos(m\varphi) - \delta\mathrm{i}\sin(m\varphi). \quad (4.7)$$

Intersections (4.6), corresponding to the forward and backward travelling waves, occur in the subcritical region ($|\Omega| < 1$) and are marked in figure 8 by open circles. The filled circles indicate the intersections of the forward and reflected waves taking place in the supercritical region ($|\Omega| > 1$).

Using the perturbation theory developed by Kirillov & Seyranian (2004, 2005) and Günther & Kirillov (2006), we find the asymptotic formula for the eigenvalues originated after the splitting of the double eigenvalues due to

interaction of the rotating string with the external loading system

$$\lambda = \lambda_{nm}^{\varepsilon\delta} - \frac{f_{nn}^{\varepsilon\varepsilon} + f_{mm}^{\delta\delta}}{2} - \frac{\epsilon_{nn}^{\varepsilon\varepsilon} + \epsilon_{mm}^{\delta\delta}}{2}$$

$$\pm \sqrt{\frac{(f_{nn}^{\varepsilon\varepsilon} - f_{22}^{\delta\delta} + \epsilon_{nn}^{\varepsilon\varepsilon} - \epsilon_{22}^{\delta\delta})^2}{4} - (f_{nm}^{\varepsilon\delta} + \varepsilon_{nm}^{\varepsilon\delta})(f_{mn}^{\delta\varepsilon} + \epsilon_{mn}^{\delta\varepsilon})}. \qquad (4.8)$$

The coefficients $f_{nm}^{\varepsilon\delta}$ are defined by

$$f_{nm}^{\varepsilon\delta} = \frac{2\lambda_{nm}^{\varepsilon\delta} \int_0^{2\pi} u_n^{\varepsilon'} \bar{u}_m^\delta \, d\varphi + 2\Omega_{nm}^{\varepsilon\delta} \int_0^{2\pi} u_n^{\varepsilon''} \bar{u}_m^\delta \, d\varphi}{2\sqrt{\int_0^{2\pi} (\lambda_{nm}^{\varepsilon\delta} u_n^\varepsilon + \Omega_{nm}^{\varepsilon\delta} u_n^{\varepsilon'}) \bar{u}_n^\varepsilon \, d\varphi \int_0^{2\pi} (\lambda_{nm}^{\varepsilon\delta} u_m^\delta + \Omega_{nm}^{\varepsilon\delta} u_m^{\delta'}) \bar{u}_m^\delta \, d\varphi}} \Delta\Omega, \qquad (4.9)$$

while the quantities $\epsilon_{nm}^{\varepsilon\delta}$ are

$$\epsilon_{nm}^{\varepsilon\delta} = \frac{(d\lambda_{nm}^{\varepsilon\delta} + k) u_n^\varepsilon(0) \bar{u}_m^\delta(0) + \mu u_n^{\varepsilon'}(0) \bar{u}_m^\delta(0)}{2\sqrt{\int_0^{2\pi} (\lambda_{nm}^{\varepsilon\delta} u_n^\varepsilon + \Omega_{nm}^{\varepsilon\delta} u_n^{\varepsilon'}) \bar{u}_n^\varepsilon \, d\varphi \int_0^{2\pi} (\lambda_{nm}^{\varepsilon\delta} u_m^\delta + \Omega_{nm}^{\varepsilon\delta} u_m^{\delta'}) \bar{u}_m^{\delta'} \, d\varphi}}, \qquad (4.10)$$

with $\Delta\Omega = \Omega - \Omega_{nm}^{\varepsilon\delta}$. Taking into account expressions (4.6) and (4.7) yields

$$\lambda = \lambda_{nm}^{\varepsilon\delta} + i\frac{\varepsilon n + \delta m}{2} \Delta\Omega + i\frac{n + m}{8\pi nm}(d\lambda_{nm}^{\varepsilon\delta} + k) + \frac{\varepsilon + \delta}{8\pi}\mu \pm \sqrt{c}, \qquad (4.11)$$

where

$$c = \left(i\frac{\varepsilon n - \delta m}{2} \Delta\Omega + i\frac{m - n}{8\pi mn}(d\lambda_{nm}^{\varepsilon\delta} + k) + \frac{\varepsilon - \delta}{8\pi}\mu \right)^2$$

$$- \frac{(d\lambda_{nm}^{\varepsilon\delta} + k - i\varepsilon n\mu)(d\lambda_{nm}^{\varepsilon\delta} + k - i\delta m\mu)}{16\pi^2 nm}. \qquad (4.12)$$

Although formula (4.11) is applicable to any node of the spectral mesh, we consider only those at $\Omega = 0$ as the most relevant to the problems of acoustics of friction. Since in this case $m = n$ and $\varepsilon = -\delta$, we find that the double eigenvalue in splits due to action of gyroscopic forces and an external spring as

$$\lambda = in + i\frac{k}{4\pi n} \pm i\sqrt{n^2\Omega^2 + \frac{k^2}{16\pi^2 n^2}}. \qquad (4.13)$$

The effect of damping and gyroscopic forces yields

$$\left(\text{Re } \lambda + \frac{d}{4\pi} \right)^2 + n^2\Omega^2 = \frac{d^2}{16\pi^2}, \qquad \text{Im } \lambda = n, \qquad (4.14)$$

$$n^2\Omega^2 - (\text{Im } \lambda - n)^2 = \frac{d^2}{16\pi^2}, \qquad \text{Re } \lambda = -\frac{d}{4\pi}, \qquad (4.15)$$

while circulatory and gyroscopic forces lead to the perturbed eigenvalues with

$$\text{Im } \lambda = n \pm \frac{1}{2\pi}\sqrt{2\pi^2 n^2\Omega^2 \pm \pi n\Omega\sqrt{4\pi^2 n^2\Omega^2 + \mu^2}}, \qquad (4.16)$$

$$\text{Re } \lambda = \pm\frac{1}{2\pi}\sqrt{-2\pi^2 n^2\Omega^2 \pm \pi n\Omega\sqrt{4\pi^2 n^2\Omega^2 + \mu^2}}. \qquad (4.17)$$
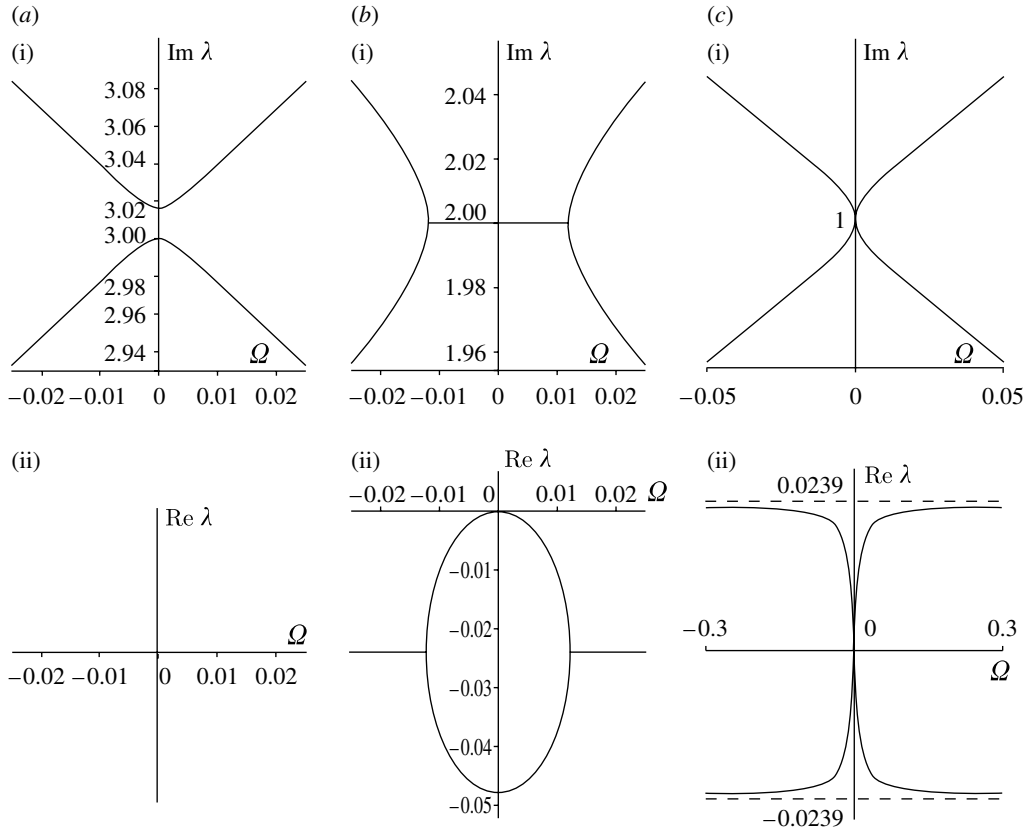
Figure 9. Deformation of the spectral mesh of the rotating string near the nodes $(0, 3)$, $(0, 2)$ and $(0, 1)$ caused by the action of the $(a(i),(ii))$ external spring with $k=0.3$, $(b(i),(ii))$ damper with $d=0.3$, and $(c(i),(ii))$ friction with $\mu=0.3$, respectively.

The lower branch of the hyperbola (4.13) passes through the node Im $\lambda=n$, while the upper one intersects the axis $\Omega=0$ at Im $\lambda=n+(k/2\pi n)$ in the plane $(\Omega, \text{Im } \lambda)$ (figure 9a). In the two-dimensional case, the reason for such a degenerate behaviour is zero eigenvalue in the matrix $\boldsymbol{K}$ of external potential forces.

The external damper creates a latent source of subcritical flutter instability exactly as it happens in two dimensions when $\boldsymbol{D}$ has one zero eigenvalue. Indeed, the bubble of instability (4.14) together with the adjacent hyperbola (4.15) is under the plane Re $\lambda=0$, touching it at the origin, as shown in figure 9b.

At a distance from the nodes, the action of gyroscopic forces and external friction deforms the spectral mesh similar to the two-dimensional case. Formal expansion of (4.17) at $\Omega\to\infty$ shows that the real parts of the perturbed eigenvalues

$$\text{Re } \lambda = \pm\frac{\mu}{4\pi}\mp\frac{\mu^3}{128\pi^3 n^2 \Omega^2} + o(\Omega^{-2}) \tag{4.18}$$

are close to the lines $\pm\mu/(4\pi)$, except for the vicinity of the node of the spectral mesh, where the real parts rapidly tend to zero as

$$\text{Re } \lambda = \pm\frac{1}{2\pi}\sqrt{\pi n\mu|\Omega|} + O(\Omega^{3/2}), \tag{4.19}$$

demonstrating the cuspidal deviation of the generic splitting picture registered in figure 9c. Expanding expression (4.16) in the vicinity of $\Omega = 0$, we find that

$$\text{Im } \lambda = n \pm \frac{1}{2\pi} \sqrt{\pi n \mu |\Omega|} + O(\Omega^{3/2}). \tag{4.20}$$

For $\Omega \to \infty$ the imaginary parts tend to $n(1 \pm \Omega)$. Thus, at $\Omega = 0$ the double eigenvalue in does not split due to non-conservative perturbation from the eyelet so that both the real and imaginary parts of the perturbed eigenvalue branches show a degenerate crossing, touching at the node $(0, n)$. In two dimensions this would correspond to the skew-symmetric matrix $\mathbf{N}$ with det $\mathbf{N} = 0$, i.e. to $\mathbf{N} \equiv 0$.

Deformation patterns of the spectral mesh obtained by the perturbation theory and shown in figure 9 qualitatively agree with the results of numerical calculations for the string (Yang & Hutton 1995) and for the disc (Xiong *et al.* 2002). They show that the perturbations from a *pointwise* external source of potential, damping and friction forces are *equally* degenerate. Even without the friction term in (4.3) the degeneracy of the model persists, as is clearly seen from the comparison of figure 9a,b with figures 2b and 3a. Similar effect was detected for the rotating disc in a pointwise frictional contact in Hutton *et al.* (1987), Xiong *et al.* (2002) and Spelsberg-Korspeter *et al.* (submitted). The pointwise type of a contact as a reason for the degeneracies in the boundary-eigenvalue problem associated with a tubular cantilever conveying fluid was discussed by Bou-Rabee *et al.* (2002).

5. Discussion: how to play a disc brake?

Supporting an attractive thesis by Chan *et al.* (1994) 'Flutter instabilities in brake systems occur primarily as a result of symmetry [breaking]; the frictional mechanism which has been the subject of much research over the past forty years is of secondary importance', the sensitivity analysis of this paper demonstrates how the nodes of the spectral mesh, situated in the subcritical range, may serve as the 'keyboard' of a rotating elastic body of revolution.

The frictional contact is a source of non-Hamiltonian and symmetry-breaking perturbations. In the vicinity of the 'keys' of the keyboard damping creates eigenvalue bubbles, which are dangerous by the ability to get positive real parts in the presence of non-conservative positional forces or even without them, if the damping is indefinite. The activated bubbles of instability cause subcritical flutter of a rotating structure, forcing it to vibrate at a frequency close to the double frequency of the node and at the angular velocity close to that of the node. The typical scenarios of eigenvalue movement due to specific structure of the matrices of dissipative and non-conservative forces, revealed in two dimensions, take place also in distributed models of a rotating circular string and a disc in a pointwise frictional contact.

An advantage of the sensitivity analysis of the spectral mesh to arbitrary perturbations is in selecting the generic behaviour of eigenvalues and thus the generic perturbations yielding flutter or stability. For example, the observed degeneracy in the movement of eigenvalues of the rotating string and disc evidences that a pointwise contact leads to the semi-definite perturbation operators that suppress generic instability mechanism behind the squeal. The

effect seems to be similar to the so-called Herrmann–Smith paradox of a beam resting on a uniform Winkler elastic foundation and loaded by a follower force (Kirillov & Seyranian 2002). Therefore, more correct description of the frictional contact would take into account the finite dimensions of the pads as well as the dependence of their characteristics on material coordinates. The size of the friction pads and their placement with respect to the rotating body should select the particular node of the spectral mesh that produces an unstable complex eigenvalue. The selection rules as well as the optimal distribution of the stiffness, damping and friction characteristics of the pads can be effectively investigated with the approach developed in this paper.

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