Re-visiting structural optimization of the Ziegler pendulum: singularities and exact optimal solutions

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Structural optimization of non-conservative systems with respect to stability criteria is a research area with important applications in fluid-structure interactions, friction-induced instabilities, and civil engineering. In contrast to optimization of conservative systems where rigorously proven optimal solutions in buckling problems have been found, for non-conservative optimization problems only numerically optimized designs were reported. The proof of optimality in the non-conservative optimization problems is a mathematical challenge related to multiple eigenvalues, singularities on the stability domain, and non-convexity of the merit functional. We present a study of the optimal mass distribution in a classical Ziegler's pendulum where local and global extrema can be found explicitly. In particular, for the undamped case, the two maxima of the critical flutter load correspond to a vanishing mass either in a joint or at the free end of the pendulum; in the minimum, the ratio of the masses is equal to the ratio of the stiffness coefficients. The role of the singularities on the stability boundary in the optimization is highlighted and extension to the damped case as well as to the case of higher degrees of freedom is discussed.

1 Structural optimization of the Ziegler's pendulum

Consider the classical Ziegler's pendulum consisting of two light and rigid rods of equal length l. The pendulum is attached to a firm basement by a viscoelastic revolute joint with the stiffness coefficient c_1 and the damping coefficient d_1. Another viscoelastic revolute joint with the stiffness coefficient c_2 and the damping coefficient d_2 connects the two rods [1]. At the second revolute joint and at the free end of the second rod the point masses m_1 and m_2 are located, respectively. The second rod is subjected to a tangential follower load P [1].

Small deviations from the vertical equilibrium for the undamped Ziegler's pendulum are described by the equation $\mathbf{M}\ddot{\mathbf{x}} + \mathbf{K}\mathbf{x} = 0$ with the mass and stiffness matrices that have the following form [1]

$$\mathbf{M} = l^2 \begin{pmatrix} m_1 + m_2 & m_2 \\ m_2 & m_2 \end{pmatrix}, \quad \mathbf{K} = \begin{pmatrix} c_1 + c_2 - Pl & Pl - c_2 \\ -c_2 & c_2 \end{pmatrix}, \tag{1}$$

where $\mathbf{x} = (\theta_1, \theta_2)^T$ is the vector consisting of small angle deviations from the vertical equilibrium position.

Calculating the characteristic equation $\det(\mathbf{M}\lambda^2 + \mathbf{K}) = 0$ for the Ziegler's pendulum without dissipation and using the Gallina criterion [2] we find a critical surface that separates flutter instability and marginal stability domains

$$p := \frac{Pl}{c_2} = 2 + \frac{1}{2}\left(\sqrt{\frac{m_1}{m_2}} \pm \sqrt{\frac{c_1}{c_2}}\right)^2 \geq 2. \tag{2}$$

The case when $c_1 = c_2 = 1$, $m_1 = 2$ and $m_2 = 1$ corresponds to the classical result of Ziegler: $p = 7/2 \pm \sqrt{2}$ [1].

The critical load (2) as a function of the masses $p = p(m_1, m_2)$ is plotted in Fig. 1(a). It is seen that the stability boundary has a self-intersection along a ray of the p-axis that starts at the Whitney umbrella singularity with the coordinates $(0, 0, 2)$ in the (m_1, m_2, p)-space. Indeed, for small absolute values of $m_1c_2 - m_2c_1$ we can expand the critical load in a series

$$p = 2 + \frac{(m_1c_2 - m_2c_1)^2}{8c_1c_2m_2^2} + O((m_1c_2 - m_2c_1)^3), \tag{3}$$

that gives an approximation to the flutter boundary in the canonical for the Whitney umbrella form $Z = X^2/Y^2$.

According to the inequality (2) the critical load is always not less than $p_0 = 2$. The minimum is reached when the masses satisfy the constraint $m_1c_2 = m_2c_1$. Note that the equal stiffness coefficients $c_1 = c_2$ imply equal masses $m_1 = m_2$. This situation corresponds to a uniformly distributed mass and stiffness in continuous systems such as the Beck's column [3–6].

Usually, in the structural optimization problems the uniformly distributed stiffness and mass are considered as the initial design that is a starting point in optimization procedures. The critical load of the optimized structure is conventionally compared to that of the same structure with the uniform mass and stiffness distributions [3–6].

Since $p(m_1, m_2)$ is a ruled surface and thus p effectively depends on the mass ratio only, it is convenient to introduce the azimuth angle α by assuming $m_1 = \cos\alpha$ and $m_2 = \sin\alpha$ and to plot the critical load as a function of α. In Fig. 1(b) the

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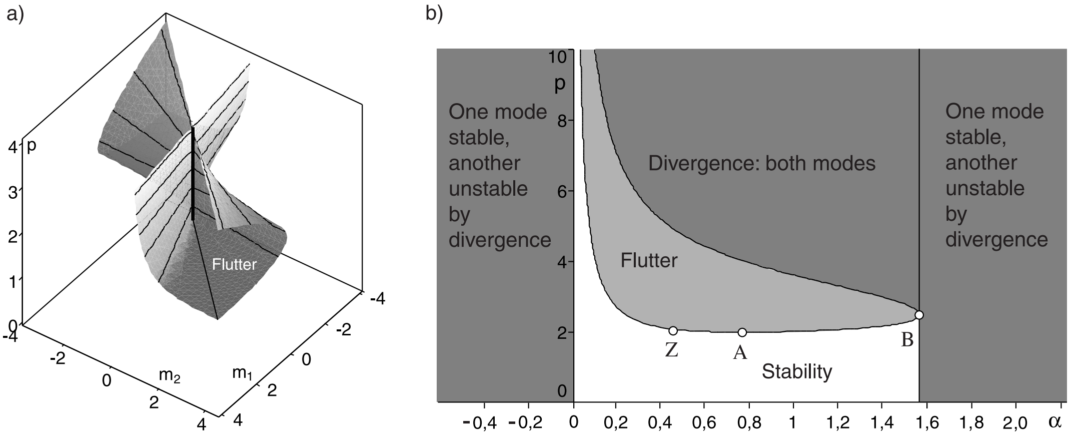

Fig. 1 Undamped 2-link Ziegler's pendulum. (a) The critical load $p(m_1, m_2)$ as a function of masses, forms a self-intersecting surface with the Whitney umbrella singularity at the point $(0, 0, 2)$ of the (m_1, m_2, p)-space (the case when $c_1 = c_2 = 1$ is shown). (b) The critical flutter load $p(\alpha)$ as a function of the azimuth angle α indicating the direction in the (m_1, m_2)-plane. The point A is an absolute minimum of the flutter load: $p_A = 2$, the point B corresponds to the local maximum: $p_B = 2 + c_1/(2c_2)$ with $m_1 = 0$, and the absolute maximum corresponds to a point C (not shown) with $p_C = +\infty$ and $m_2 = 0$. The point Z corresponds to the Ziegler's original design: $m_1/m_2 = 2$.

curves $p = p(\alpha)$ bound the flutter domain shown in the light gray. When α tends to zero, which corresponds to the vanishing mass m_2, the critical load increases to infinity. When α tends to $\frac{\pi}{2}$ and, correspondingly, the mass m_1 is vanishing, then the critical flutter load increases to the value $p_B = 2 + \frac{1}{2}\frac{c_1}{c_2}$. At the point B in the stability diagram of Fig. 1(b) the flutter boundary has a vertical tangent, which is a typical phenomenon in non-conservative optimization [7].

To summarize, the popular initial design corresponding to uniformly distributed mass and stiffness turns out to give an absolute minimum to the critical flutter load of the Ziegler's pendulum. The critical flutter load attains its local maximum, p_B, for $m_1 = 0$ at the singular cuspidal point B of the stability boundary where the flutter domain has a vertical tangent and touches the boundary of the divergence domain. Note that in [8] a local extremum of the flutter load for the free-free beam carrying a point mass was found to be at the cuspidal point on the flutter boundary too. The global maximum of the critical flutter load for the undamped Ziegler's pendulum is at infinity when $m_2 = 0$.

The global maximum corresponds to a vanishing mass at the free end of the column which qualitatively is in agreement with the numerically found optimized designs of the Beck's column available in the literature [3–6]. Indeed, all known optimized designs of the Beck's column are characterized by the vanishing cross-sections at the free end. Moreover, the gradients of the critical flutter load with respect to the mass or stiffness distribution of the Beck column are large, which is, again, in qualitative agreement with our stability diagram of Fig. 1(b) . The most interesting is the fact that with the increase of the critical flutter load the higher and higher modes were reported to be involved into the coupling that indicates the onset of flutter [3–6]. Our simple model shows that this phenomenon seems to be natural for the optimal design that causes the degeneracy in the mass matrix that gives rise to the critical frequency that increases without bounds.

The same qualitative picture we observed in the case of an m-link Ziegler pendulum [9, 10]. Since the system is finite-dimensional and contains the finite number of control parameters with the clear physical meaning, the locations of the singularities corresponding to multiple eigenvalues can easily be found numerically with the high accuracy. In the vicinity of such points where at least three pure imaginary eigenvalues couple, the question 'Should low-order models be believed' [11] makes sense because here a one more degree of freedom is crucial for the correct solution.

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